ФИЗИЧЕСКИЙ ФАКУЛЬТЕТ МГУ ИМЕНИ М.В. ЛОМОНОСОВА

teach-in
ЛЕКЦИИ УЧЕНЫХ МГУ

ФИЗИКА НЕЙТРИНО. ЧАСТЬ 1

СТУДЕНИКИН АЛЕКСАНДР ИВАНОВИЧ

ФИЗФАК МГУ

Contents

Lecture 1. What is the main property of neutrino

The structure of effective Hamiltonian

In order to get neutrinos we can observe decay of a muon or a neutron:

$$n \to p^- + e^+ + \tilde{\nu}_e, \tag{1.1}$$

$$\mu^- \to e^- + \tilde{\nu}_e + \nu_\mu \tag{1.2}$$

During our lectures we shall derive the lifetime of a neutron. This value will depend on G_F and $\bar{\alpha}$. It took decades for people to observe different weak interaction of particles to determine this type of effective Hamiltonian:

$$H = \frac{G_F}{\sqrt{2}}\left(\Psi^+, \gamma_\mu(1 + \bar{\alpha}\gamma_5)\Psi_n \times \Psi_e^+\gamma^\mu(1 + \gamma_5)\Psi_\nu\right), \tag{1.3}$$

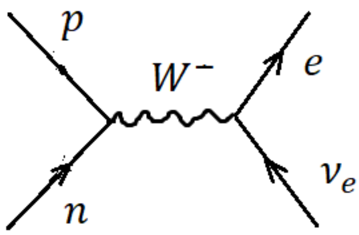
and also $\bar{\alpha}$ was fixed to 1.25.

The Standard Model. Fermi (effective) theory

The Standard Model of electroweak interactions includes all the information above but in this case we know that the neutron decay process is due W-boson propagation. So the G_F constant is inverse proportional to the mass of the W-boson:

$$G_F \sim \frac{1}{{M_W}^2} \tag{1.4}$$

If we are not interested in the quark structure of a proton and a neutron the Feynman diagram of the neutron decay will look like



Picture 1.1. The Feynman diagram of the neutron decay.

In the Fermi theory instead of the W-boson there was a Fermi-constant G_F because the Fermi theory have no idea about any mediators.

The Fermi theory is often called the Effective theory because it is not possible to have a perturbation expansion within this theory unlike the Standard Model. In this sense the Standard Model is a universal theory. And in the first order of perturbation series expansion of the Standard Model we exactly produce what was proposed by Fermi but of course if the energies are not extremely high.

Mass of neutrino. Dirac equation

The main property of neutrino particle is the mass. Nowadays we have some information about the mass of neutrino but not final and not exact. We know for sure that the mass of neutrino is not zero $m_\nu \neq 0$ but we don't know the real value. The situation is very tricky because neutrinos are not exact particles but combination of real particles called neutrino mass states:

$$\nu_e = a_1\nu_1 + b_1\nu_2 + c_1\nu_3 \tag{1.5}$$

These particles have real masses: m_1, m_2, m_3. Other types of neutrino ν_μ and ν_τ are also combinations of the mass states but with different numbers:

$$\nu_\mu = a_2\nu_1 + b_2\nu_2 + c_2\nu_3, \tag{1.6}$$

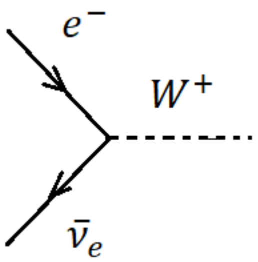
$$\nu_\tau = a_3\nu_1 + b_3\nu_2 + c_3\nu_3 \tag{1.7}$$

So ν_1, ν_2, ν_3 are real particles with real masses. It means that for each of these three particles we can write the equation of motion in vacuum (Dirac equation for neutrino mass states):

$$(p - m_\nu)\Psi_i(x) = 0, \qquad i = 1,2,3 \tag{1.8}$$

Standard Model Lagrangian. Different types of neutrinos

But the Standard Model Lagrangian $\mathcal{L}^{SM}$ is determined in terms of flavor states not neutrino mass states. Let's see how we can distinguish three types of flavor neutrino by particular terms in the Standard Model lagrangian. The typical term in $\mathcal{L}^{SM}$ which enable us to distinguish different types of flavor neutrino:

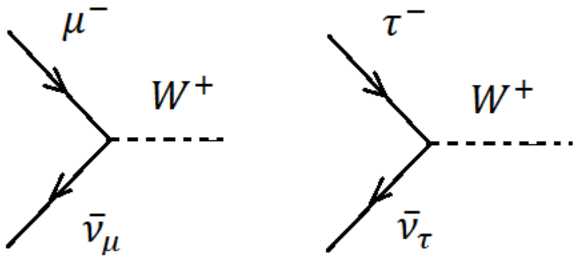


Picture 1.2. The typical term of the SM Lagrangian.

This Lagrangian should be used when you are considering the process of appearance of neutrino and also when you are dealing with neutrino detection. But when neutrinos propagate from the source to the detector you should forget about flavor neutrinos and distribute the composed ν_e, ν_μ, ν_τ particles into ν_1, ν_2, ν_3 to know how to write the equation of motion. If you agree that neutrino mass states are not equal to each other in general $m_1 \neq m_2 \neq m_3$ you can see that flavor neutrinos have no well-defined mass. We can't write the Dirac equation for

ν_e, ν_μ, ν_τ neutrinos. They are composed of ν_1, ν_2, ν_3. And when these conglomerate starts moving it will be separated in space in time. You see that in general the neutrino is not a normal particle but a kind of phenomena. When you consider the interaction of neutrinos you should use flavor neutrino states and The Standard Model theory of interaction. When you try to investigate the propagation of neutrinos of different flavors you should always consider the neutrino mass states.

Back to the question about how we distinguish different types of flavor neutrino, we can write the similar diagrams for ν_μ and ν_τ:



Picture 1.3. Other terms of the SM Lagrangian.

There is no mass term in $\mathcal{L}^{SM}$ like $m_\alpha \bar{\Psi}_{\nu_\alpha} \Psi_{\nu_\alpha}, \alpha = e, \mu, \tau$.

Neutrinos behave like particles well defined by the Standard Model lagrangian but it is not possible even to describe how the flavor neutrino propagates because there is no equation of motion. The main problem of neutrino study is neutrino mass. The best present limit of effective neutrino mass obtained from tritium decay experiments:

$$m_\nu^{eff} < 0.8\ eV \tag{1.9}$$

From (1.9) we can have some more direct information about particular constraints of m_1, m_2, m_3 masses. Anyway we remember that m_ν is not zero. There is a set of experiments when people measure fluxes of neutrinos from the Sun. In the middle of 60s the Nobel Prize winner Davis for the first time observed the flux of neutrinos from the Sun and got three times less value then it was expected based on the Standard Model calculations. So Davis detected only 1/3 from the amount that was expected if the Sun is working as it is really working.

There are other theoretical and experimental indications that something is not very simple with neutrinos. Under the pressure of this experimental data decomposition of flavor neutrinos into neutrino mass states were proposed. For the first time this idea was formulated in short paper by Bruno Pontecorvo, distinguished Soviet Union and Russian scientist. He moved from Italy to the USSR together with his family in 1955. From 1966 to 1988 he was the staff member of The Faculty of Physics of the Moscow State University. In 1957 he proposed that neutrinos are not simple particles. At that time only neutrino ν and antineutrino $\bar{\nu}$ were known. The second flavor muon neutrino ν_μ was discovered only in 1962. In his paper

Pontecorvo wrote that probably neutrino mass is not zero and that neutrino is a mixture of neutrinos and antineutrinos. In the next sentence he presented the final result of his idea that after some distance the flux of neutrinos would be composed in equal amount of neutrinos and antineutrinos. Pontecorvo's idea was used to explain why in the middle of 60s Davis measured 1/3 of the amount of solar neutrinos in respect to what was expected if the model of the Sun has been working perfectly. After the experimental discovery of 1962 was made Japanese scientists Sakata, Maki and Nakagawa introduced the Pontecorvo's idea of mixture and second flavor of neutrino but didn't change the second part of Pontecorvo's prediction about evolution of neutrino flux in space and time. They fixed it and put it just as it was. Later in 1967 Pontecorvo and Gribov considered the evolution in space and time of mixed states of neutrinos. They for the first time have derived the formula that described the probability to observe the particular flavor of neutrino flux after it has been traveled some distance in time. That was the final result in the theory of mixing and oscillation. Nowadays the matrix of mixing has been written and called the Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS matrix).

Information of neutrino masses from observation of electron energy spectrum in Beta-decay

Let's discuss the process in which the upper bound on effective neutrino mass can be achieved. We will consider the energy spectrum of electrons that appear in β-decay. Initial parent nucleus decays into a daughter nucleus, electron and antineutrino:

$$(A,Z) \to (A,Z+1) + e^{+} + \tilde{\nu}_e, \tag{1.10}$$

$$n \to p^{-} + e^{+} + \tilde{\nu}_e \tag{1.11}$$

It is important to know that

$$\frac{\Delta}{m_e} = \frac{m_n - m_p}{m_e} \cong 2.5\,, \tag{1.12}$$

where $m_e \cong 0.5\, MeV$. The maximum energy that can be distributed between electron and antineutrino in (1.11) process is $\Delta \sim 1.3\, MeV$. If you consider a nuclear decay due to the defect of masses the energy deposit will be different. We suppose that the mass of neutrino is very small because the existed limit of the effective neutrino mass is much less than 1 MeV. We can write the energy of a nuclear decay:

$$\begin{aligned} Q_{(m_\nu=0)} &= [m(A,Z) - Zm_e]c^2 - [m(A,Z+1) - (Z+1)m_e + m_e]c^2 \\ &= [m(A,Z) - m(A,Z+1)]c^2 \end{aligned} \tag{1.13}$$

(Further we shall make calculations in the system where $\frac{h}{2\pi} = c = 1$ but now we prefer the presence of speed of light in our formula.) This energy can be distributed among the three particles that appear in the decay. If we neglect the mass of neutrino and account for that the electron mass is much less than the mass of the daughter nucleus then we can say that all the energy is related to the electron. If the neutrino mass is not zero then obviously the energy spectrum of electrons will be modified in respect to the neutrino mass case. That is the main idea of experiments to obtain information of an effective neutrino mass – measuring the energy spectrums of electrons in the β-decay.

KATRIN experiment about effective mass of neutrino

KATRIN collaboration (Germany) in 2019 announced that the effective neutrino mass

$$\text{KATRIN 2019:} \quad m_\nu^{eff} < 1\ eV \tag{1.14}$$

Nowadays in 2023 they've improved their result to slightly lower limit

$$\text{KATRIN 2023:} \quad m_\nu^{eff} < 0.8\ eV \tag{1.15}$$

The same experiment was produced about 20 years ago here in Russia (Troizk) in the Institute of Nuclear Research of Russian Academy of Sciences (INR RAS):

$$\text{INR RAS:} \quad m_\nu^{eff} < 2.05\ eV \tag{1.16}$$

A little bit later there was an experiment in Mainz:

$$\text{Mainz:} \quad m_\nu^{eff} < 2.2\ eV \tag{1.17}$$

The results of INR RAS and Mainz experiments results were combined to

$$\text{INR RAS, Mainz:} \quad m_\nu^{eff} < 2\ eV \tag{1.18}$$

Let's study the process of tritium decay. It decays into helium, electron and antineutrino:

$${}^3_1H \to {}^3_2H + e^+ + \tilde{\nu}_e \tag{1.19}$$

We will make some very simple estimation of energy and momentum characteristics of emitted electrons. If the initial nucleus is at rest and we account for the fact that the mass of electron is much less that the mass of daughter nucleus $m_e \ll m(A, Z+1)$ we can say that the energy of electron is equal to the energy deposited in the process:

$$E_e = Q \approx m(A,Z) - m(A,Z+1) \tag{1.20}$$

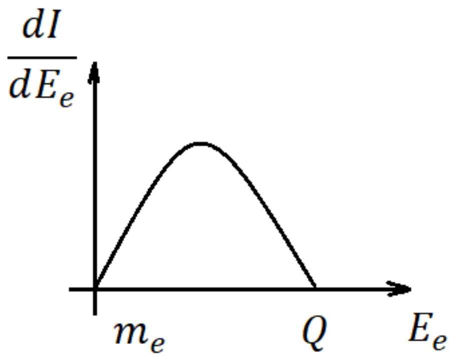
We can write that

$$\frac{E_e}{E_p} = \frac{m_p}{m_e} \tag{1.21}$$

Of course we deal with energies at which particles are not relativistic. In case of tritium decay it is known that

$$Q \approx 18.6\ eV \tag{1.22}$$

Let's recall the energy spectrum of electrons in case that there is no neutrino (pic. 1.4). So the spectrum is monochromatic.



Picture 1.4. Energy spectrum of electrons.

We should do more exact calculations and derive an exact formula for the energy spectrum of electrons. Before we start this I would like to mention the student of Department of Theoretical Physics Alexey Lokhov who worked for many years in our group, got his diploma and defended the PhD dissertation here. He worked on the theory of neutrino electromagnetic properties. After he finished his education he got a permanent position in Institute of Nuclear Research and was very important participant of the mass experiment. He proposed a very sophisticated method of experimental data analysis. About four years ago he became the member of KATRIN experiment and now for three years he is in Germany. He was one of the scientists who did an analysis of KATRIN experimental data and announced it at the international neutrino conference in Kyoto in Japan.

The probability of Beta-decay of tritium

We will write the probability of β-decay of tritium using so-called golden rule of Fermi:

$$\frac{dW}{dt} = \frac{2\pi}{\hbar}\left|\langle f|H_\beta|i\rangle\right|^2 \frac{dn}{dE}, \quad (1.23)$$

$|i\rangle -$ the initial state, $\langle f| -$ the final state of particles in the process, $H_\beta -$ effective Hamiltonian, $\frac{dn}{dE} -$ the density of the final state over the energy formed by dn_e and dn_ν because the daughter nucleus is too heavy and don't move. This equation is obtained in the perturbation series expansion in the first order within the Fermi theory. The matrix element of the interaction Hamiltonian $M_{fi} = \langle f|H_\beta|i\rangle$ is squared in our expression (1.23) and determined by the effective Hamiltonian .

At first we suppose that the mass of neutrino is zero $m_\nu = 0$. Assume that the final state of electrons is determined by

$$\frac{dn}{dE} = \frac{dn_e dn_\nu}{dE} \quad (1.24)$$

Our goal is to derive the energy spectrum of electrons:

$$N(p_e)dp_e = \frac{2\pi}{\hbar}\left|\langle f|H_\beta|i\rangle\right|^2 \frac{dn}{dE} \tag{1.25}$$

Let’s calculate the faith space of electrons which is the combination of coordinates x and momentum p spaces. We will address to the Heisenberg uncertainty principle and recall that the uncertainty in the coordinates and in the corresponding momentum couldn’t be less than the plank constant:

$$\Delta x\Delta p \geq \hbar \tag{1.26}$$

This is a minimal value of the faith space of electrons. If we have 3 coordinates and 3 momentums then the faith space is six dimensional: $\Delta x\Delta y\Delta z\Delta p_x\Delta p_y\Delta p_z$. We neglect the spin properties. Let's consider the electron that has been radiated in the decay and localize in some volume in space V with the momentum in the interval $(p_e, p_e + dp_e)$. The momentum shell envelope:

$$d^3p_e = 4\pi p_e^2 dp_e \tag{1.27}$$

The number of different electron states in the volume Vd^3p_e is

$$dn_e = \frac{V4\pi p_e^2 dp_e}{\hbar^3} \tag{1.28}$$

The number of different neutrino states dn_ν you should calculate by yourself.

Lecture 2. Neutrino mass. The end of introduction

The energy spectrum of electrons depends on neutrino mass

Last time we discussed the aspects of the problem of neutrino mass. We remember that neutrino is not a usual particle so we should distinguish behavior of neutrino when it is produced or detected and we should consider flavor states and when it is propagating we should forget about flavors and deal with the mass states. We started the discussion on how is it possible to get information about the value of neutrino mass. Normally we should think about the masses of the mass states because flavor neutrinos have no exactly determined masses. Nevertheless in many processes flavor neutrino can be considered as an elementary particle with some effective mass, in many processes where mixing and oscillations are not important. There is a set of experiments that can be called the direct search of neutrino mass. The aim of these experiments is to measure or at least to limit the effective mass. Nowadays there are no experiments where the effective mass has been measured but there are experiments where the bound on effective mass has been obtained. Let's continue our discussion of some theoretical approach which will open for us how these kinds of experiments are organized.

Last time we discussed the β-decay of tritium. The decay of one neutron in tritium looks like

$$n \to p^{+} + e^{-} + \tilde{\nu}_e \tag{2.1}$$

Tritium decays into helium, electron and antineutrino:

$$^3_1H \to {}^3_2He + e^{-} + \tilde{\nu}_e \tag{2.2}$$

We considered the case when the masses of the initial and daughter nuclei are much bigger than the possible masses of neutrino. We started to calculate the energy spectrum of electrons which depends on effective neutrino mass. We used the Fermi theory of weak interactions and so-called Fermi golden rule for the probability of β-decay of tritium:

$$\frac{dW}{dt} = \frac{2\pi}{\hbar}\left|\langle f|H_\beta|i\rangle\right|^2 \frac{dn}{dE}, \tag{2.3}$$

$|i\rangle -$ the initial state $({}^3_1H)$, $\langle f| -$ the final state of particles $({}^3_2He, e, \tilde{\nu}_e)$, $H_\beta -$ effective Hamiltonian, $\frac{dn}{dE} -$ the density of the final state over the energy. There are a number of electrons emitted in the β-decay with a particular momentum p_e in the interval dp_e:

$$N(p_e)dp_e = \frac{2\pi}{\hbar}\left|\langle f|H_\beta|i\rangle\right|^2 \frac{dn}{dE}, \tag{2.4}$$

$\frac{dn}{dE} -$ the final states number density in the unit interval dE. We can expect that this value (2.4) will depend on the mass of neutrino.

Heisenberg's uncertainty principle

From the Heisenberg's uncertainty principle it follows that the six dimensional faith space given by 2 sets of coordinates and corresponding momentums couldn't be less than the cubed plank constant:

$$(\Delta x \Delta p_x, \Delta y \Delta p_y, \Delta z \Delta p_z) \geq \hbar^3 \quad (2.5)$$

Let's consider the electron localized in some volume in space V with the momentum in the range $(p_e, p_e + dp_e)$. The momentum shell envelope is

$$d^3 p_e = 4\pi p_e^2 dp_e \quad (2.6)$$

The amount of different electron states in the volume of faith space $V d^3 p_e$ is

$$dn_e = \frac{V 4\pi p_e^2 dp_e}{\hbar^3} \quad (2.7)$$

The contribution to the final phase space of the final particles

Now let's include the contribution to the final phase space of the final particles also from neutrinos. The total amount of states is a composition of states corresponding to electrons and neutrinos:

$$dn = dn_e dn_\nu \quad (2.8)$$

The distributions for these particles are independent. So we can write the same (2.7) formula for neutrinos. The total number density of the final states over the energy will be

$$\frac{dn}{dE} = \frac{16\pi^2}{\hbar^6} V^2 p_e^2 p_\nu^2 \frac{dp_e dp_\nu}{dE} \quad (2.9)$$

How does the distribution depend on the mass of neutrino

We should start with the total energy deposited in the process:

$$Q = E \quad (2.10)$$

We can write the energy conservation law that the energy is distributed between electrons, neutrinos and the daughter nucleus:

$$E = E_e + E_\nu + E_{(Z+1)} \quad (2.11)$$

We suppose that the daughter nucleus is much heavier than two other particles in the final state: $m_e, m_\nu \ll m_{Z+1}$. It means that the daughter nucleus is not moving and

$$E_{(Z+1)} \cong 0 \quad (2.12)$$

Let's consider the case when the mass of neutrino is zero $m_\nu = 0$. It follows from (2.11) and (2.12) that

$$E_e + E_\nu = E \quad (2.13)$$

The neutrino momentum equals to

$$p_\nu = \frac{E_\nu}{c} = \frac{E - E_e}{c} \quad (2.14)$$

For the fixed electron energy we get that

$$\frac{dp_\nu}{dE} = \frac{1}{c} \quad (2.15)$$

We can sum up all our considerations and get the momentum spectrum of electrons from β-decay:

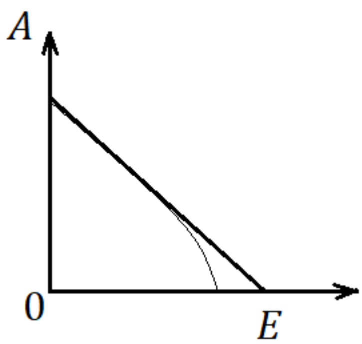
$$N(p_e)dp_e = const\left|\langle f|H_\beta|i\rangle\right|^2 p_e^2(E - E_e)^2 dp_e \quad (2.16)$$

Coulomb interaction of electrons. Curie plot

The formula (2.16) can be illustrated by a very simple figure that is called the Curie plot. It is important to note that there is a Coulomb interaction of electrons which modify the momentum spectrum by some special function $F(Z, E_e)$. It can be calculated for each particular decay. The Curie plot is the dependence of some effective value

$$A = \sqrt{\frac{N(p_e)}{p_e^2 F(Z, E_e)}} = B(E - E_e) \quad (2.17)$$

The dependence $A(E_e)$ is just a straight line (pic. 2.1). In case of massless neutrino we have very primitive behavior of $A(E_e)$.



Picture 2.1. The Curie plot.

Without any calculations we can answer the question what will happen with the Curie plot when the neutrino mass is not zero. The line on the plot will end before E because the part of available energy will be used to generate the mass of neutrino. It means that if we are looking for the effect of neutrino mass we should examine the end point of the electron spectrum. The problem is that closer to the end point less particles you have.

The case of the mass is not zero

Let's write an exact formula which is the basis for analyzing results of β-decay experiments. Consider the case of the mass is not zero. As we were saying above the energy E will be shifted to $E - m_\nu c^2$. We have to analyze again $\frac{dp_\nu}{dE}$ in case $m_\nu \neq 0$. The energy of neutrino equals to

$$E_\nu = \sqrt{p_\nu^2 c^2 + m_\nu^2 c^4} - m_\nu c^2 \Rightarrow \tag{2.18}$$

$$E_\nu^2 + m_\nu^2 c^4 + 2E_\nu m_\nu c^2 = p_\nu^2 c^2 + m_\nu^2 c^4 \Rightarrow \tag{2.19}$$

$$p_\nu = \frac{1}{c}\sqrt{E(E_\nu + 2m_\nu c^2)} = \frac{1}{c}\sqrt{(E-E_e)^2 - (m_\nu c^2)^2} \Rightarrow \tag{2.20}$$

$$p_\nu^2 \frac{dp_\nu}{dE} = \frac{1}{2} p_\nu \frac{dp_\nu^2}{dE} = \frac{1}{c^3}(E-E_e)^2 \sqrt{1 - \left(\frac{m_\nu c^2}{E-E_e}\right)^2} \tag{2.21}$$

So for the momentum spectrum of electrons from β-decay we get

$$N(p_e)dp_e = B^2 F(Z, E_e) p_e^2 (E-E_e)^2 \sqrt{1 - \left(\frac{m_\nu c^2}{E-E_e}\right)^2}\, dp_e \tag{2.22}$$

It shows exactly how the momentum spectrum depends on the mass of neutrino. If we just put $m_\nu = 0$ in (2.22) we will reproduce (2.16). For the Curie plot we'll get

$$A|_{m_\nu \neq 0} = \sqrt{\frac{N(p_e)}{p_e^2 F(Z, E_e)}} = B(E-E_e)\left[1 - \left(\frac{m_\nu c^2}{E-E_e}\right)^2\right]^{1/4} \tag{2.23}$$

The maximum energy available for electrons becomes obviously smaller. The most spectacular effect happens at the energies near the end point of the spectrum. For small energies modification of the plot is insignificant.

Experiments

First of all the tritium decay has very good energy for these kinds of experiments and secondly the structure of the nucleus is very simple. There were two experiments one of them took place not far from Moscow in Troizk by Institute for Nuclear Research of RAC. The similar experiment was conducted in Meinz in Germany. Both these experiments set the following limit for the effective neutrino mass:

$$m_\nu^{eff} \leq 2\ eV \tag{2.24}$$

This result was obtained about 15 years ago. The best limit for the effective neutrino mass was obtained in KATRIN experiment in Germany. The idea of this experiment is the same but the scale is much bigger. In 2019 in Japan, Kyoto at the Neutrino conference they announced new bound on effective neutrino mass:

$$m_\nu^{eff} \leq 1\ eV \tag{2.25}$$

Last year (2023) they've achieved the best limit:

$$m_\nu^{eff} \leq 0.8\ eV \tag{2.26}$$

I would like to mention one of our former students of Department of theoretical physics of the Moscow State University Alexey Lokhov who worked for many years in our group, got his diploma and defended the PhD dissertation here. After he finished his education he got a permanent position in Institute of Nuclear Research and was very important participant of the neutrino mass experiment. He became the member of KATRIN experiment. He was one of the scientists who did an analysis of KATRIN experimental data and announced it at the international neutrino conference in Kyoto in Japan.

The effect of mixing and oscillations of neutrinos

There are two motivations to include these effects in neutrino physics. We should recall that the idea of mixing and oscillations was first announced by Bruno Pontecorvo, distinguished Soviet Union and Russian scientist who have made a huge contribution to neutrino physics. Bruno Pontecorvo (1913-1993) was born in Italy in 1913. In 1950 he with his family moved to Soviet Union. In 1930s he was a student of Fermi, one of the fathers of American nuclear project. It was the most distinguished and important group in the particle physics. Pontecorvo have never participated in so-called Manhattan experiment but he was very close to it. Till the very end of his life he was the staff member of The Joint Institute for Nuclear Research in Dubna. Also from 1966 he was the staff member of Faculty of physics of Moscow State University, he chaired the Department of particle physics.

In 1957 Pontecorvo published two very short papers. At that time it was not clear whether neutrinos are massless or massive particles. From the Dirac theory you know that if the particle is massless it is quite enough to have two component wave function but if the mass is not zero you need four components. Pontecorvo wrote: "It is not clear whether we should use two or four component wave function to describe neutrinos" In other words it was not clear for him whether neutrino is massive or massless particle. At that time only one type of neutrino was known and when he was spoken about two different neutrinos he meant neutrino ν and antineutrino $\bar{\nu}$. If the four component wave function should be used then two neutrinos are not different particles but they are particle mixtures. So Pontecorvo introduced that if the mass of neutrino is not zero then there is a mixing between different types of neutrinos (1957). Moun neutrino was discovered only in 1962 and tau neutrino – in 2000.

In the same papers Pontecorvo wrote as a result that on a certain distance from the reactor the number of neutrinos and antineutrinos will be equal:

$$\frac{N(\nu)}{N(\bar{\nu})} = 1 \tag{2.27}$$

It is the result of oscillations. Later in 1969 together with Griov he derived an exact formula for oscillations.

In the 60s-70s there were two problems, first of them related to the solar neutrinos. The flux of the solar neutrinos measured in the experiment by Davis divided by the flux of the solar neutrinos calculated according to the Solar Standard Model is only one third:

$$\frac{\Phi_{\nu_\odot}^{exp}}{\Phi_{\nu_\odot}^{SSM}} = \frac{1}{3} \tag{2.28}$$

If we do not incorporate the effects of mixing and oscillations we will never explain why Davis measured less than it was predicted by the Standard Model of the Sun.

Another problem is the atmospheric neutrino problem. The experimental flux of muon neutrinos divided by the flux of electron neutrinos that expected by the Standard Model is less than 2:

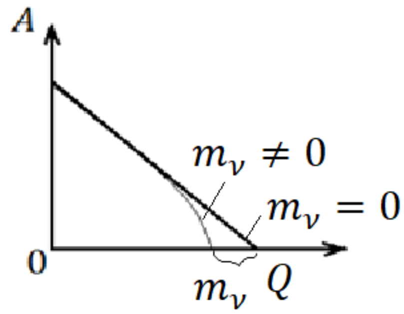
$$\left.\frac{\Phi_{\nu_\mu}^{exp}}{\Phi_{\nu_e}^{SM}}\right|_{atm} < 2 \tag{2.29}$$

There were two particular experiments which finally confirmed that in order to solve the solar and atmospheric neutrino problems we should agree that there is mixing and oscillations of neutrino. The results of these experiments were awarded a Nobel Prize in 2015 for discovery of neutrino oscillations that confirms that neutrino mass is not zero. First experiment with solar neutrinos was conducted by Sudbury Neutrino Observatory in Canada (Arthur McDonald), the results were published in 2002. Second experiment with atmospheric neutrinos was conducted in Japan, it is called Supper-Kamiokande, the results were announced in 1998.

Lecture 3. Neutrino mixing and oscillations

Repetition of the previous lecture. Calculations. Task

The most important question of neutrino physics is the problem of neutrino mass. We know that the mass of neutrino is for sure not zero. There are two problems in neutrino physics. First problem related to the solar neutrinos. The flux of the solar neutrinos measured in the experiment turned out to be only one third from the flux of the solar neutrinos calculated according to the Solar Standard Model. Another problem related to the atmospheric neutrinos. The experimental flux of muon neutrinos divided by the flux of electron neutrinos that expected by the Standard Model turned out to be less than expected 2. These two problems can be solved only due to effects of mixing and oscillations. That's why we can say for sure that the mass of neutrino is not zero but we do not know the exact value. One of the possible ways to measure the effective neutrino mass is to observe the end point of the electron spectrum that appear in the β-decay.



Picture 3.1. The Curie plot.

We draw a Curie plot as the dependence of value $A\sim\frac{dN}{dE}$ from electron energy E_e(pic. 3.1). In case of massless neutrino it is just a straight line. When the neutrino mass is not zero the line on the plot will end before E because the part of available energy will be used to generate the mass of neutrino. For the momentum spectrum of electrons from β-decay we got

$$N(p_e)dp_e = B^2F(Z,E_e)p_e^2(E-E_e)^2\sqrt{1-\left(\frac{m_\nu c^2}{E-E_e}\right)^2}\,dp_e \qquad (3.1)$$

Task.

Please show that the number of electrons in the very end of the spectrum $n(Q-\Delta E)$ is proportional to

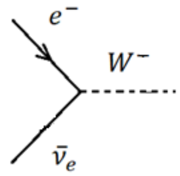
$$n(Q-\Delta E)\sim\left(\frac{\Delta E}{Q}\right)^3, \qquad (3.2)$$

$\Delta E = Q - E_e$.

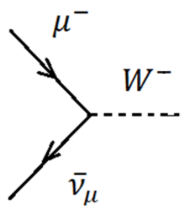
Example: $Q \cong 18.6\ keV, \Delta E \cong 20\ eV$.
If you remember the best upper limit on m_ν^{eff} is 0.8 eV. It has been obtained by the KATRIN collaboration starting from 2019 till 2023.

The most simple case of neutrino mixing and oscillations

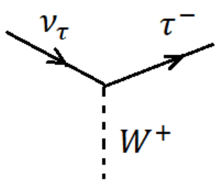
Let's consider the electron ν_e and muon ν_μ neutrinos. We can specify these two particles using the Standard Model Lagrangian. There is particular term in $\mathcal{L}_{SM}$:



Once you see electron you can suppose that electron neutrino is participating in this process. Similar diagram is used for identification of the muon neutrino:



For instance there is an experiment called OPERA in which muon neutrino produced in CERN travel about 735 kilometers to Gran Sasso. After 10 years of running this experiment they've succeeded to detect tau leptons which means that tau neutrino arrived to Gran Sasso:



Previously we discussed flavor neutrinos. There is another vision on this phenomenon which is about neutrino mass states. Neutrino mass states ν_1, ν_2 have well determined masses m_1 and m_2. For these neutrinos we can solve the problem of propagation and write the Dirac equation

$$(\hat{p} - m_1)\Psi_{\nu_1} = 0, \qquad (3.3)$$

$$(\hat{p} - m_2)\Psi_{\nu_2} = 0 \qquad (3.4)$$

But we don't know how these types of neutrinos are produced and detected because the SM Lagrangian which enables us to calculate the probability of appearance and detection processes is written in terms of flavor neutrinos.

We observe some kind of dualism: when neutrinos produced and detected we should describe them as flavor neutrinos but when they propagate we should use mass states. Just to make everything together we should propose that the flavor neutrinos are combination of the mass states. There was no other way to solve two major neutrino problems: the solar neutrino problem and the atmospheric neutrino problem. As you remember in 1957 Pontecorvo predicted that neutrino is a mixture between neutrino and antineutrino

The neutrino flavor and the mass states

Let's introduce the flavor neutrino like a vector with two components: $\nu^f = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$ which can be whether electron or muon neutrino and neutrino mass states $\nu^p = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ with the masses m_1 and m_2, where "p" means "physical". We should notice that

$$\nu^f \neq \nu^p \tag{3.5}$$

We suppose that the flavor neutrino is a combination of the physical neutrino:

$$\nu^f = U\nu^p, \tag{3.6}$$

where $U -$ is the mixing matrix. To find the solution of the solar and atmospheric neutrino problems we should solve the problem of evolution of flavor neutrinos. We shall write the evolution equation of Schredinger for physical neutrinos:

$$i\frac{d}{dt}\nu^p = H\nu^p \tag{3.7}$$

Knowing equation (3.7) and using the transformation (3.6) we can express the physical neutrinos in terms of flavor neutrinos and derive the evolution equation for flavor neutrinos. The Hamiltonian in (3.7) is

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \tag{3.8}$$

$$E_\alpha|_{\alpha=1,2} = \sqrt{m_\alpha^2 + \vec{p}_\alpha^{\,2}}, \tag{3.9}$$

where we suggest that all neutrinos in any experiments up to now are relativistic particles. (There is only one source of non-relativistic neutrinos from very early moments of the universe, they are called cosmological neutrinos. But nobody has yet observed them.) We can expand the energy (3.9) and take into account only the first two terms that will be

$$E_\alpha = |\vec{p}_\alpha| + \frac{1}{2}\frac{m_\alpha^2}{|\vec{p}_\alpha|} \tag{3.10}$$

We suppose that the momentums of two neutrinos are equal:

$$p_1 \cong p_2 \tag{3.11}$$

We notice some kind of paradox two particles with different masses having equal momentums. But we should remember that masses m_α are much less than the energies E_α: $m_\alpha \ll E_\alpha$. The Hamiltonian will look like

$$H = |\vec{p}| + \frac{1}{2|\vec{p}|}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \tag{3.12}$$

Let's introduce the well-known Pauli matrixes

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.13}$$

We can rewrite (3.12) in a following way

$$H = \left(|\vec{p}| + \frac{m_1^2 + m_2^2}{4|\vec{p}|}\right)\sigma_0 - \frac{\Delta m^2}{4|\vec{p}|}\sigma_3, \tag{3.14}$$

where $m_2^2 - m_1^2 = \Delta m^2$- the mass square difference of neutrinos. This value turns out to be very important in theory and during interpretation of experimental data.

The evolution of flavor neutrinos

We know the Hamiltonian for evolution of the mass states and the relation between the mass and flavor states. So using the evolution equation for the mass states we can derive the evolution equation for the flavor states. We introduced the flavor neutrinos as

$$\nu^f = U\nu^p, \tag{3.15}$$

where $U -$ unitary matrix, it means $U^+ = U^{-1}$. From (3.15) we can express the physical states as

$$\nu^p = U^+\nu^f \tag{3.16}$$

We can put this relation (3.16) into initial equation of evolution for physical state. Also we shall use the following very important property that evolution in space can be related to evolution in time

$$\frac{d}{dx} \leftrightarrow \frac{d}{dt} \tag{3.17}$$

So the evolution equation for flavor states looks like

$$i\frac{d}{dx}(U^+\nu^f) = HU^+\nu^f \Rightarrow \tag{3.18}$$

$$i\frac{d}{dx}\nu^f = UHU^+\nu^f \tag{3.19}$$

The mixing matrix

We should somehow parameterize the matrix U. It is possible to show that only one parameter is enough in this case when we have two mass states and two flavor states. So we will introduce the neutrino mixing angle θ and express the mixing matrix as

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3.20}$$

So we can write that

$$\begin{cases} \nu_e = \nu_1\cos\theta + \nu_2\sin\theta \\ \nu_\mu = -\nu_1\sin\theta + \nu_2\cos\theta \end{cases} \tag{3.21}$$

It's obvious form (3.21) that if ν_1 and ν_2 have well determined masses than ν_e and ν_μ for sure haven't. We cannot write separate Dirac equations for electron and muon neutrinos. Of course if we consider tau neutrino we should include ν_3 with mass m_3 and there will be another mass square differences Δm_{12}^2 and Δm_{13}^2.

Let's find the Hamiltonian of evolution of flavor neutrinos which we will call Hamiltonian prime:

$$H' = UHU^+ \tag{3.22}$$

To continue we should recall the following statements:

$$U\sigma_0 U^+ = I, \tag{3.23}$$

$$U\sigma_3U^+ = \begin{Bmatrix} U^+ = U^{*T} \\ s \equiv \sin\theta \\ c \equiv \cos\theta \end{Bmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$
$$= -\begin{pmatrix} -c & s \\ s & c \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = -\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \tag{3.24}$$

Finally

$$H' = \left(|\vec{p}| + \frac{m_1^2 + m_2^2}{4|\vec{p}|}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta m^2}{4|\vec{p}|}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \tag{3.25}$$

As soon as the Hamiltonian prime doesn't depend on x we can easily integrate the evolution equation for flavor neutrinos and get the solution:

$$\nu^f(x) = e^{-iH'x}\nu^f(0), \tag{3.26}$$

$x -$ coordinate along the neutrino propagation.

The probability of oscillations between different flavor neutrino states

Now we should calculate the probability of neutrino transition from one flavor state to another. Notice that the part of Hamiltonian which is proportional to σ_0 doesn't change the initial flavor. So we can write that

$$H' = \frac{\Delta m^2}{4E}(\sigma_1\sin 2\theta - \sigma_3\cos 2\theta), \tag{3.27}$$

where $E \cong |\vec{p}|$. Using $(3.26), (3.27)$ we can write the solution for the flavor neutrinos:

$$\nu^f(x) = e^{-i\frac{\Delta m^2}{4E}x(\sigma_1 \sin 2\theta - \sigma_3 \cos 2\theta)}\nu^f(0) \tag{3.28}$$

We see that the matrices σ_1 and σ_3 enter the exponent function, so we should expand it in a row. First let's recall the following statements:

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots, \tag{3.29}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots, \tag{3.30}$$

$$\sin x = x - \frac{x^3}{3!} + \cdots, \tag{3.31}$$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{3.32}$$

Using above equations we can write that

$$e^{i\alpha\sigma_i} = \sum_{n=0}^{\infty} \frac{(i)^{2n}\sigma_i^{2n}}{(2n)!}\alpha^{2n} + \sum_{n=0}^{\infty} \frac{i^{(2n+1)}\sigma_i^{2n+1}}{(2n+1)!}\alpha^{2n+1} \Rightarrow \tag{3.33}$$

$$e^{i\alpha\vec{\sigma}} = I\cos\alpha + i\vec{\sigma}\sin\alpha, \tag{3.34}$$

$$e^{-i\alpha\vec{\sigma}} = I\cos\alpha - i\vec{\sigma}\sin\alpha \tag{3.35}$$

Now we are only one step ahead to write the exact formula for probability, for example, to observe the muon neutrino in the initial flux of electron neutrinos or the probability of oscillations. We can write the solution for the flavor neutrinos:

$$\nu^f(x) = e^{-i\frac{\Delta m^2}{4E}x(\sigma_1 \sin 2\theta - \sigma_3 \cos 2\theta)}\nu^f(0)$$
$$= \left[\cos\frac{\Delta m^2}{4E}x - i(\sigma_1 \sin 2\theta - \sigma_3 \cos 2\theta)\sin\frac{\Delta m^2}{4E}x\right]\nu^f(0) \tag{3.36}$$

Now we can put forward the problem what is the probability to observe different flavor of neutrino in respect to the flavor that was produced after the flux of neutrinos has traveled a distance from 0 to x. The probability to observe muon neutrino ν_μ in the initial flux of electron neutrinos ν_e or the probability of oscillations $P_{\nu_e \to \nu_\mu}$. For example, as you remember Davis measured only $1/3$ from the amount of electron neutrinos predicted by Standard Model. It means that $1/3$ of electron neutrinos was transformed to muon and tau neutrinos during travel. Of course we consider a simplified version with only two flavors and two mass states. The solution for the solar neutrino problem mostly can be reproduced by transition of electron neutrinos to muon neutrinos. But the solution of the atmospheric neutrino problem is mostly given by the transition of muon neutrinos to tau neutrinos. We see that in both cases only two types of neutrino involved. So according to the rules of quantum mechanics

$$P_{\nu_e\to\nu_\mu}(x) = \left|\langle\nu_\mu|\nu_e(x)\rangle\right|^2 = \left|(0,1)\sigma_1 \sin 2\theta \sin\frac{\Delta m^2}{4E}x\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \sin^2 2\theta \sin^2\frac{\Delta m^2}{4E}x\,, \quad (3.37)$$

where $\sin^2 2\theta -$ the amplitude of oscillations. We should recall that the mass of neutrino is not zero because if it is than

$$\Delta m^2 = m_2^2 - m_1^2 = 0 \quad (3.38)$$

And also we should notice that not only the mass of neutrino is not zero $m_\nu \neq 0$ but the masses of neutrino mass states are not equal to each other $m_1 \neq m_2$.

This is the most spectacular phenomenon of neutrino physics because it is not within The Standard Model. According to The Standard Model the mass of neutrino equals to zero and there is no mixing and oscillations and there is no difference between flavor and mass states. There is no solution of the solar and atmospheric neutrino problems if you are confined within The Standard Model. There is no other observed particle that describes beyond The Standard Model.

Lecture 4. The problem of mixing and oscillations of three neutrinos

Main points of the previous lecture

We were considering the effects of neutrino mixing and oscillations. We have been forced to introduce these phenomena under the pressure of very strong experimental facts from astrophysics due to solar and atmospheric neutrinos. So we have been obliged to include these effects in addition to The Standard Model where the mass of neutrino is exactly zero and where are no mixing and oscillations. We introduced the mass states of neutrino $\nu_\alpha, \alpha = 1{,}2{,}3$ with corresponding masses m_α. For these neutrino mass states we can solve the problem of propagation and write the Dirac equations:

$$(\hat{p} - m_\alpha)\Psi_\alpha(x) = 0, \tag{4.1}$$

where $\hat{p} = \gamma_\mu p^\mu$. But we don't know how the mass states interact with other particles or, in other words, we don't know the Standard Model Lagrangian of interaction. In The Standard Model we have to incorporate other types of neutrinos or so-called flavor states of neutrino $\nu_f, f = e, \mu, \tau$. They enter into the Standard Model Lagrangian $\mathcal{L}_{SM}$ as fields. Flavor neutrinos are not accessible for the problem of propagation because they have no masses. We suppose that these two different approaches of description this unusual phenomenon should be combined. That's why we introduced the mixing effect, so all flavor neutrinos are the mixtures of the mass states. We are going to consider the simple case when there are only two flavor and mass states. So we can introduce the mixing matrix:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.2}$$

The flavor neutrinos can be expressed through the mass states:

$$\begin{cases} \nu_e = \nu_1\cos\theta + \nu_2\sin\theta \\ \nu_\mu = -\nu_1\sin\theta + \nu_2\cos\theta \end{cases} \tag{4.3}$$

Now we can calculate the probability of conversion of one type of flavor neutrino to another. We postulate that the Hamiltonian of evolution of mass states is given by the following formula:

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \tag{4.4}$$

$$E_\alpha|_{\alpha=1,2} = \sqrt{m_\alpha^2 + \vec{p}_\alpha^2} \cong |\vec{p}_\alpha| + \frac{1}{2}\frac{m_\alpha^2}{|\vec{p}_\alpha|} \tag{4.5}$$

We consider relativistic neutrinos than $E_\alpha \cong |\vec{p}_\alpha|$ and evolution in space corresponds to evolution in time. We can find the Hamiltonian of evolution of flavor neutrinos H'. Than we can calculate the probability to detect muon neutrino in the initial electron neutrinos flux:

$$P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\theta \sin^2 \frac{\Delta m^2}{4E} x, \tag{4.6}$$

where $\Delta m^2 = m_2^2 - m_1^2$ is the mass square difference. There are two key parameters that determine the oscillations probability: $\sin^2 2\theta$ and Δm^2. All these ideas were proposed by Bruno Pontecorvo in 1957. After the second type of neutrino was discovered in 1962 these ideas were used by Japanese group of Maki, Nakagawa and Sakata. They've included the mixing between two other types of neutrino $\nu_e \leftrightarrow \nu_\mu$ but not the oscillations. In 1969 Pontecorvo and Gribov calculated the probability of oscillations.

Oscillations probability. Oscillations length

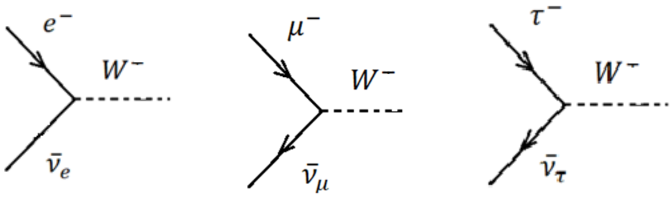
Let's introduce such value as the oscillation length:

$$L = \frac{4\pi E}{\Delta m^2} \tag{4.7}$$

If $\Delta m^2 = m_2^2 - m_1^2 \to 0$ than there are no oscillations. Therefore the masses of the mass states are not zero $m_{1,2} \neq 0$ and they are not equal to each other $m_1 \neq m_2$.

The problem of mixing and oscillations of three neutrinos

Let's go further and discuss more general case with three types of neutrinos ν_e, ν_μ, ν_τ. There are particular terms in $\mathcal{L}_{SM}$ for each type:



Picture 4.1. The $\mathcal{L}_{SM}$ terms.

If neutrino of the particular flavor arrived to the detector you will detect the corresponding charge lepton. We also will consider three mass states of neutrino ν_1, ν_2, ν_3 with corresponding masses m_1, m_2, m_3 for which we can write the equations of motion:

$$(\hat{p} - m_\alpha)\Psi_\alpha(x) = 0, \qquad \alpha = 1,2,3 \tag{4.8}$$

We suppose that the neutrino was born with the particular coordinate $x = 0$ at the initial moment $t = 0$ as the neutrino with the particular flavor $\nu \equiv \nu_f, f = e, \mu, \tau$. Considering the effect of propagation of the mass states and the effect of mixing in the particular point (x, t) we can write that

$$|\nu_f(x,t)\rangle = \sum_{i=1}^{3} U_{fi} e^{-iE_i t} e^{ip_i x} |\nu_i\rangle, \tag{4.9}$$

where U_{fi} − a mixing matrix, i − an index of a mass state. We considered above that neutrino propagates as a plain wave with a particular energy and momentum. We should also recall that there is a relativistic relation between energy and momentum for each neutrino mass state:

$$E_i = \sqrt{|\vec{p}_i|^2 + m_i^2} \tag{4.10}$$

In addition to (4.9) we should mention that each mass state is a superposition of flavor states:

$$|\nu_i\rangle = \sum_k U_{ki}|\nu_k\rangle, \tag{4.11}$$

where k − an index of a flavor state. Let's put (4.11) into (4.9):

$$|\nu_f(x,t)\rangle = \sum_k \sum_{i=1}^{3} U_{ki}U_{fi}e^{-iE_it}e^{ip_ix}|\nu_k\rangle \tag{4.12}$$

Equal energy prescription and equal momentum prescription

From this point to go further we should choose one of two prescriptions. We weather suppose that the energies of two neutrinos are equal or their momentums. The result will be the same.

1) Equal energy prescription

We will use the decomposition of the particular momentum:

$$p_i = \sqrt{E_i^2 - m_i^2} \cong E_i - \frac{1}{2}\frac{m_i^2}{E_i} \tag{4.13}$$

Therefore

$$-iE_it + ip_ix \cong -iE_it + iE_ix - \frac{i}{2}\frac{m_i^2}{E_i}x = -\frac{i}{2}\frac{m_i^2}{E_i}x = -i\frac{m_i^2}{2E}t \tag{4.14}$$

2) Equal momentum prescription

We will use the decomposition of the particular energy:

$$E_i = \sqrt{p_i^2 + m_i^2} \cong p_i + \frac{1}{2}\frac{m_i^2}{p_i} \tag{4.15}$$

Therefore

$$-iE_it + ip_ix \cong -ip_it - \frac{i}{2}\frac{m_i^2}{p_i}t + ip_ix = -\frac{i}{2}\frac{m_i^2}{p_i}t = -i\frac{m_i^2}{2p}t \tag{4.16}$$

The probability of neutrino oscillations in case of equal momentum prescription

We can write the probability of oscillations in case when there are three types of neutrino using the equal momentum prescription:

$$P_{f\to l}(t) = \left|A_{f\to l}\right|^2 = \left|\sum_{\substack{k \\ flavor}} \sum_{\substack{i \\ mass}} U_{ki} U_{fi} e^{-\frac{i}{2}\frac{m_i^2}{p_i}t} \langle \nu_l | \nu_k \rangle\right|^2 = \left|\sum_{\substack{i \\ mass}} U_{li} U_{fi} e^{-\frac{i}{2}\frac{m_i^2}{p_i}t} \langle \nu_l | \nu_k \rangle\right|^2$$

$$= \sum_j \sum_i U_{li} U_{fi} U_{lj} U_{fj} e^{-\frac{i}{2}\left(\frac{m_i^2}{p_i} - \frac{m_j^2}{p_j}\right)t} = \sum_j \sum_i U_{li} U_{fi} U_{lj} U_{fj} e^{-\frac{i}{2}\frac{\Delta m_{ij}^2}{p}t}$$

$$= \sum_{i=j} U_{li}^2 U_{fi}^2 + 2\sum_{i<j} U_{li} U_{fi} U_{lj} U_{fj} \cos\left(2\pi \frac{t}{\tau_{ij}}\right), \qquad (4.17)$$

where $f, l = e, \mu, \tau$; $\langle \nu_l | \nu_k \rangle = \delta_{lk}$; $\Delta m_{ij}^2 = m_i^2 - m_j^2$; $\tau_{ij} = 2\pi \frac{2p}{\left|\Delta m_{ij}^2\right|}$.

The probability of neutrino oscillations in case of equal energy prescription

Using the equal energy prescription we will get the following formula:

$$P_{f\to l}(x) = \sum_{i=j} U_{li}^2 U_{fi}^2 + 2\sum_{i<j} U_{li} U_{fi} U_{lj} U_{fj} \cos\left(2\pi \frac{x}{l_{ij}}\right), \qquad (4.18)$$

where $l_{ij} = 2\pi \frac{2E}{\left|\Delta m_{ij}^2\right|}$ – the length of oscillations.

Now let's discuss how to use these formulas to interpret the real experiment's results. Let's consider for instance the experiment with the solar neutrinos $\nu_{\odot}$. The point is that the neutrinos are produced in the inner part of the Sun that is about 10% of the radius of the Sun. It means that we do not exactly know from which point neutrino travels to the detector. In order to use our idealistic formulas we should average them over the distance of neutrino travel. So what is measured is the average value of the probability over time:

$$\langle P_{f\to l}(t) \rangle_T = \frac{1}{T}\int_0^T P_{f\to l}(t)dt = \sum_i U_{li}^2 U_{fi}^2 + \frac{2}{T}\int_0^T \sum_{i<j} U_{li} U_{fi} U_{lj} U_{fj} \cos\left(2\pi \frac{x}{l_{ij}}\right) dt \Bigg|_{T\to\infty}$$

$$= \sum_i U_{li}^2 U_{fi}^2 \qquad (4.19)$$

The first conclusion is that probability depends only on the mixing matrix. Secondly there is no dependence on mass square difference.

The maximal degree of suppression of initial flavor neutrino for N neutrinos

Suppose there are N flavor neutrinos $\nu_f, f = 1, \dots, N$ and also N mass states of neutrinos $\nu_i, i = 1, \dots, N$. What is the maximum degree of suppression of the initial flavor neutrino (after the neutrino flux has traveled a certain distance)? The problem is to find the optimal structure of the mixing matrix that provides the maximum possible suppression in case of N flavor neutrinos and N mass states. You should recall your knowledge from the first year of the mathematical analysis course and consider the conditional extremum of a function. The answer is that the maximum suppression equals to $1/_N$. As you remember the suppression of the electron solar neutrinos according to results of the experiment conducted by Davis was $1/_3$.

Lecture 5. Phenomenon of neutrino mixing

Repetition of the previous lecture. Solution of the problem. Sterile neutrinos

Last time we have derived the expression for the probability of oscillations and have introduced such characteristics as the mixing angle θ and the mass square difference Δm^2. We considered the case when there are two flavors and two mass states. And as a result the question had been asked: what is the maximum degree of suppression of the initial flavor neutrino after the neutrino flux has traveled a certain distance from the source to the detector?

The solution:

We suppose there are N flavor neutrinos ν_f: $|\nu_f\rangle$, $f = e, \mu, \tau, \dots, N$ and also N mass states of neutrinos ν_p: $|\nu_i\rangle$, $i = 1,2,3, \dots, N$.

Let's make a comment about the number of flavor states according to the last experimental data. Many years ago in LEP the Z-boson was observed. The probability of Z-boson decay $Z^0 \to \nu_f \bar{\nu}_f$ was measured. From experimental observation it follows that the number of neutrino flavors equals $N_f = 3 \pm \Delta$. It means that now it is well confirmed that there are only three neutrino flavor. Some physicists consider the fourth and the fifth flavors but they should be sterile than. Sterile neutrinos ν_s exist without interaction with other particles but they participate in mixing. So in addition to flavor neutrinos $|\nu_f\rangle$, $f = e, \mu, \tau$ we have the sterile neutrinos $|\nu_s^i\rangle$, $i = 1,2,3, \dots$. But we won't consider sterile neutrinos in this task. So the basic result is that we have only three active flavor states.

The probability of the transition of the flavor 1 type neutrino to the flavor 2 type neutrino during the time t

Each of the flavor states is a superposition of the mass states:

$$|\nu_f\rangle = \sum_i U_{fi} |\nu_i\rangle \tag{5.1}$$

The probability of the transition of neutrino with one flavor f_1 to another f_2 during time interval t or distance x:

$$P_{f_1 \to f_2}(x) = \sum_{i=j} U_{f_1 i}^2 U_{f_2 j}^2 + 2 \sum_{i<j} U_{f_1 i} U_{f_2 i} U_{f_1 j} U_{f_2 j} \cos\left(2\pi \frac{x}{l_{ij}}\right), \tag{5.2}$$

$$l_{ij} = 2\pi \frac{2E}{|\Delta m_{ij}|^2} \tag{5.3}$$

In order to use this formula we should average it over distance or time because the starting point for each particular neutrino is not fixed. Since we are interested in the survival probability in addition to probability averaged over time we also have to add the following condition that $f_1 = f_2 = e$. Therefore the average survival probability is

$$P_{\nu_e\nu_e}(t) = \sum_{i=1}^{N} |U_{ei}|^4 \tag{5.4}$$

The minimum of the probability under the condition that neutrinos do not escape and evaporate. The method of Lagrange multipliers

Now we have to find the minimum of $P_{\nu_e\nu_e}(t)$ under the condition that neutrinos do not escape and evaporate on the path from the source to the detector. This condition could be written as

$$\sum_{i=1}^{N} |U_{ei}|^2 = 1 \tag{5.5}$$

We have a typical task of finding an extremum with a condition. The solution can be achieved by using the Lagrange multipliers. According to this method we have to find the minimum of the following functional:

$$F = \sum_{i=1}^{N} |U_{ei}|^4 + \lambda \left(\sum_{i=1}^{N} |U_{ei}|^2 - 1 \right) \tag{5.6}$$

We have to solve the following equation:

$$\frac{\partial F}{\partial U_{ei}} = 0 \Rightarrow \tag{5.7}$$

$$4|U_{ei}|^3 + 2\lambda |U_{ei}| = 0 \tag{5.8}$$

We suppose that $N = 3, i = 1,2,3$. Obviously we are not interested in the solution without mixing so we exclude it

$$|U_{ei}| = 0 \tag{5.9}$$

Therefore

$$|U_{ei}|^2 = -\frac{\lambda}{2}, i = 1,2,3 \tag{5.10}$$

According to the condition (5.5) we get

$$|U_{ei}|^2 = \frac{1}{N} \Rightarrow \tag{5.11}$$

$$P_{\nu_e\nu_e}(t)\Big|_{min} = \left(\sum_{i=1}^{N} |U_{ei}|^4 \right)\Bigg|_{min} = N\frac{1}{N^2} = \frac{1}{N} \tag{5.12}$$

So we have obtained that the maximum available degree of suppression in the case of N flavor and N mass states of neutrinos is $1/_N$. It means that if $N = 3$ the minimum available degree of suppression will be $^1/_3$. As you remember Davis has measured exactly this value. Earlier we discussed the additional sterile neutrinos. If the fourth sterile neutrino is present we can expect that the suppression will be $^1/_4$. So if people see that the suppression is bigger than expected $^1/_3$ it means that there are some additional sterile neutrino.

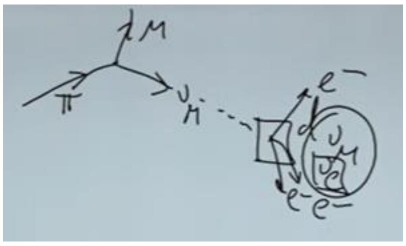
The limits on the appearance of oscillation phenomenon are due to coherence

You may notice that the theory that we develop for the description of neutrino mixing is very similar with the optics when we consider the interference phenomenon. Of course we discuss an ideal picture, considering neutrinos as waves in reality will lead to some principal constraints. These constraints come from the lack of coherence between two waves describing two oscillating neutrinos.

Let's consider an experiment where two flavor neutrinos are produced in π meson decay

$$\pi^+ \to \mu^+ + \nu_\mu \to \nu_e \tag{5.13}$$

We are interested how muon neutrino oscillates to electron neutrino. We would like to derive the coherence constraints for the appearance of neutrino oscillation phenomenon.



Picture 5.1. An experiment visualization.

Muon neutrinos produced in π meson decay travel to the detector. If we found electrons in the detector it means that some of muon neutrinos have oscillated to electron neutrinos (pic. 5.1).

Let's suppose that we are well equipped enough that we can exactly measure the momentums and energies $(P_{\pi,\mu}; E_{\pi,\mu})$ of particles π^+ and μ^+. It means that we can calculate all neutrino's characteristics from the energy and momentum conservation. Then we suppose that the emitted muon neutrino is a superposition of two mass states with the masses m_1, m_2:

$$\nu_\mu = a_1\nu_1 + a_2\nu_2 \tag{5.14}$$

If we know everything about π^+ and μ^+ we can say for sure in which particular mass state the flavor neutrino appears.

Now let's consider the problem of the coherence of two neutrino waves and derive the constraints for observation of neutrino oscillation pattern in an experiment. To solve this problem we have to consider the difference of phases of two neutrinos ν_1 and ν_2 after they have traveled the distance x from the point where they were produced to the detector.

$$\varphi(x,t) = (E_1 - E_2)t - (\vec{p}_1 - \vec{p}_2)\vec{r} \tag{5.15}$$

We will use the relativistic expressions:

$$p_\mu = (p_0, \vec{p}), p_0^2 = \vec{p}^2 + m^2 \tag{5.16}$$

We would like to calculate the phase difference and then determine the condition of coherence that the phase difference is smaller than some basic number (1 or 2π).

The energy momentum conservation

We have the following equation

$$p_\pi = p_\nu + p_\mu, \;\; \nu = \nu_1, \nu_2 \Rightarrow \tag{5.17}$$

$$(p_\pi - p_\nu)^2 = p_\mu^2 \Rightarrow \tag{5.18}$$

$$m_\pi^2 + m_\nu^2 - 2(p_\pi, p_\nu) = m_\mu^2, \;\; \nu = \nu_1, \nu_2 \Rightarrow \tag{5.19}$$

$$m_{\nu_1}^2 - m_{\nu_2}^2 - 2(p_\pi, p_{\nu_1}) + 2(p_\pi, p_{\nu_2}) = 0 \tag{5.20}$$

Considering the pion as nonrelativistic particle we get

$$(p_\pi, p_\nu) = E_\pi E_\nu - (\vec{p}_\pi, \vec{p}_\nu) \cong E_\pi E_\nu, \nu = \nu_1, \nu_2 \tag{5.21}$$

From (5.20) and (5.21) it follows that

$$E_{\nu_1} - E_{\nu_2} = \frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2E_\pi} \tag{5.22}$$

Further we have to consider the second contribution to the phase difference (5.15):

$$(\vec{p}_1 - \vec{p}_2)\vec{r} \equiv (p_1 - p_2)x \tag{5.23}$$

We know that

$$\left|\vec{p}_{\nu_{1,2}}\right|^2 = E_{\nu_{1,2}}^2 - m_{\nu_{1,2}}^2 \Rightarrow \tag{5.24}$$

$$\left|\vec{p}_{\nu_1}\right|^2 - \left|\vec{p}_{\nu_2}\right|^2 = E_{\nu_1}^2 - E_{\nu_2}^2 - m_{\nu_1}^2 + m_{\nu_2}^2 = \left(E_{\nu_1} - E_{\nu_2}\right)\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2E_\pi} - \left(m_{\nu_1}^2 - m_{\nu_2}^2\right)$$
$$= \frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2E_\pi}\left(E_{\nu_1} + E_{\nu_2} - 2E_\pi\right) \tag{5.25}$$

From (5.25) it follows that

$$p_1 - p_2 = \frac{m_1^2 - m_2^2}{2E_\pi}\frac{(E_1 + E_2 - 2E_\pi)}{p_1 + p_2} \tag{5.26}$$

Finally, the phase difference is

$$\varphi(x,t) = (E_1 - E_2)t - (p_1 - p_2)x$$
$$= \frac{m_1^2 - m_2^2}{2E_\pi}t + \frac{m_1^2 - m_2^2}{E_1 + E_2}t - \frac{m_1^2 - m_2^2}{E_1 + E_2}t - (p_1 - p_2)x$$

$$= \frac{m_1^2 - m_2^2}{E_1 + E_2} t + \frac{m_1^2 - m_2^2}{2E_\pi (E_1 + E_2)} (E_1 + E_2 - 2E_\pi) t - (p_1 - p_2) x$$
$$= \frac{m_1^2 - m_2^2}{E_1 + E_2} t + (p_1 - p_2) \left[\frac{p_1 + p_2}{E_1 + E_2} t - x \right] \quad (5.27)$$

Observations

Let's consider the interpretation of this formula and make the following theoretical observations. First of all we can conclude that the second term is negligible for relativistic neutrinos ν_1, ν_2 and can be excluded because

$$x = v_0 t = \frac{p_1 + p_2}{E_1 + E_2} t, \quad (5.28)$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \gamma = \frac{E}{m} \Rightarrow \quad (5.29)$$

$$\beta^2 = \frac{p_0^2 - m^2}{p_0^2} = \frac{|\vec{p}|^2}{p_0^2} \quad (5.30)$$

Therefore

$$\varphi(x, t) \sim \frac{\Delta m^2}{2E} t, \Delta m^2 = m_1^2 - m_2^2 \quad (5.31)$$

Now we can calculate

$$\delta\varphi = \delta \left(\frac{\Delta m^2}{2E} \right) t = \frac{\Delta m^2}{2} \delta \left(\frac{1}{E} \right) t \quad (5.32)$$

Using the Heisenberg uncertainty principle we get that

$$\delta E \sim \delta p \sim \frac{1}{\delta x}, \quad (5.33)$$

$\delta x -$ the uncertainty of the point where neutrino was produced in the source. Also we can write that

$$\delta \left(\frac{1}{p} \right) \sim \frac{1}{p^2} \delta p \quad (5.34)$$

Using (5.33), (5.34) we finish our formula for $\delta\varphi$:

$$\delta\varphi = \frac{\Delta m^2}{2p^2} \delta p \, x \quad (5.35)$$

The criteria of coherence

Let's introduce the coherence length L_{coh} and say that $x = L_{coh}$ when the difference of phases is small:

$$\delta\varphi < 2\pi \, (1) \quad (5.36)$$

We can calculate the maximum number of maximums or minimums in the oscillation pattern:

$$N_{max} = \frac{L_{coh}}{L_{12}} = \frac{2p^2(2\pi)}{\Delta m^2 \delta p}\frac{\Delta m^2}{4\pi p} = \frac{p}{\delta p} \sim p\delta x, \tag{5.37}$$

$L_{12} = \frac{4\pi p}{\Delta m^2}-$ the oscillation length or the spatial difference between two minimums or maximums. We also used the assumption that $E \approx p$. Usually in experiments where oscillations are observed $N_{max} \gg 1$ and we can see that the coherence condition is fulfilled.

The statement of impossibility of observing two peaks in the momentum distribution corresponding to the appearance of neutrinos with masses m_1 and m_2

Let’s show that it is not possible to simultaneously observe oscillations and two peaks in the momentum distribution corresponding to appearance of neutrinos with masses m_1 and m_2. Again we suppose that very precise experiment allows us to determine the energies and momentums of pion π^+ and muon μ^+. Knowing this we can say whether neutrino of mass m_1 or m_2 appears. If we measured the energy and momentum of muons we shall see two peaks in the spectrum of muons corresponding to ν_1 or ν_2. In this case there is no room for neutrino oscillation to be observed because the coherence criteria cannot be fulfilled. It is not possible to achieve both goals in one experiment. Once you try to determine the mass state there is no oscillation pattern, once there are oscillations you will never see in each particular mass state neutrino has been observed.

Let's do some simple mathematical calculations. Firstly, more precise you manage to determine the equations of motion for initial particles less accurate you determine their location. And if you don’t know the exact location you will never fulfill the coherence condition. The kinematic determination of the emitted neutrino is based on the well-known relation between the energy and the momentum:

$$m_\nu^2 = E_\nu^2 - p_\nu^2 \tag{5.38}$$

In order to distinguish neutrino mass state we have to consider the arrow in the mass determination:

$$\Delta(m_\nu^2) = \sqrt{(2E_\nu)^2(\Delta E_\nu)^2 + (2p_\nu)^2(\Delta p_\nu)^2} \tag{5.39}$$

We also have to add that neutrinos ν_1 and ν_2 can be distinguished only if the following condition is valid:

$$(\Delta m_\nu)^2 < m_1^2 - m_2^2 \tag{5.40}$$

Than we can say that we are interested only second part under the square root in (5.39). So the condition to distinguish the mass state is

$$\sqrt{(2p_\nu)^2(\Delta p_\nu)^2} < \Delta m^2 \tag{5.41}$$

But we know that the uncertainty in momentum is inverse proportional to uncertainty in the coordinate:

$$\Delta p_\nu \sim \frac{1}{\Delta x} \Rightarrow \tag{5.42}$$

$$\Delta x > \frac{2p_\nu}{\Delta m^2} \sim L_{12} \tag{5.43}$$

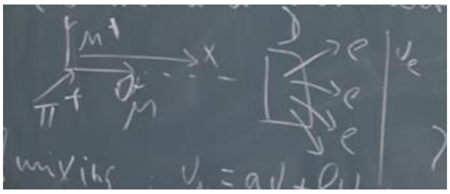
So the uncertainty in the location of the each particular emitted neutrino is bigger than the oscillation length. It means that there is no possibility to observe the oscillation pattern.

All these considerations put an inherent limit on a plan of experiments for observation of neutrino oscillations. It depends on the conditions when neutrinos were produced and it also depends on how accurate you can estimate the starting position of each neutrino.

Lecture 6. The coherence of neutrino states

We are going to discuss very important issue related to neutrino oscillations that is the coherence of neutrino states. We will derive the condition under which the phenomena of mixing and oscillations may appear. There are inherent limitations for these phenomena to appear due to uncertainties in the production region of neutrinos and as a result there is an uncertainty of the length that neutrinos propagate from the source to the detector. This consideration put an inherent limit on the probability to observe the neutrino oscillation phenomenon.

The coherence in neutrino oscillations



Picture 6.1. π^+-decay experiment

We will consider the neutrinos produced in the decay of π^+ meson with appearance of muon μ^+ and muon neutrino ν_μ (pic. 6.1). Neutrinos propagate and arrive to the detector. If there is a room for neutrino oscillation phenomenon we shall see electrons that will indicate that part of muon neutrinos have been transferred to electron neutrinos. We suppose that there is mixing in the initial point, it means that

$$\nu_\mu = a_1\nu_1 + a_2\nu_2 \tag{6.1}$$

If we carefully measure the state of motion of charged particles we can understand in which mass state neutrino appears. The initial point $t = 0$ the wave function of neutrino is

$$|\nu(x,0)\rangle = \sum_{\alpha=1,2} U_{\mu\alpha}|\nu_\alpha(x,0)\rangle = \sum_{\alpha=1,2} U_{\mu\alpha} \exp i(\vec{p}_\alpha\vec{x})\,|\nu_\alpha\rangle \tag{6.2}$$

In some later point t we have

$$|\nu(x,t)\rangle = \sum_{\alpha=1,2} U_{\mu\alpha} e^{i\vec{p}_\alpha\vec{x}} e^{-iE_\alpha t}|\nu_\alpha\rangle \tag{6.3}$$

We would like to answer the question about what is the phase difference between two mass states arriving to the detector:

$$\varphi(x,t) = (E_1 - E_2)t - (\vec{p}_1 - \vec{p}_2)\vec{r} \tag{6.4}$$

Then we should examine the uncertainty in the phase difference $\Delta\varphi(x,t)$. Obviously from our experience of wave phenomena in optics we know that if this uncertainty in the phase difference between two waves is big enough there will be no room for oscillations. The uncertainty $\Delta\varphi$ will be determined by the uncertainty in the production point of neutrinos in the source.

Of course we should start with the law of energy and momentum conservation:

$$p_\pi = p_\nu + p_\mu, \tag{6.5}$$

where

$$p_{\pi,\mu,\nu} = \left(E_{\pi,\mu,\nu}, \vec{p}_{\pi,\mu,\nu}\right), \nu = \nu_1, \nu_2 \tag{6.6}$$

Therefore

$$(p_\pi - p_\nu)^2 = p_\mu^2 \Rightarrow \tag{6.7}$$

$$m_\pi^2 + m_\nu^2 - 2(\vec{p}_\pi, \vec{p}_\nu) = m_\mu^2 \Rightarrow \tag{6.8}$$

$$m_{\nu_1}^2 - m_{\nu_2}^2 - 2\left(\vec{p}_\pi, \vec{p}_{\nu_1}\right) + 2\left(\vec{p}_\pi, \vec{p}_{\nu_2}\right) = 0 \tag{6.9}$$

Considering the pion as nonrelativistic particle, we will get

$$(\vec{p}_\pi, \vec{p}_\nu) = E_\pi E_\nu - \cancel{(\vec{p}_\pi, \vec{p}_\nu)} \cong E_\pi E_\nu, \tag{6.10}$$

So from the energy and momentum conservation it follows that

$$E_{\nu_1} - E_{\nu_2} = \frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2E_\pi} \tag{6.11}$$

Let's consider the second term in the phase difference:

$$(\vec{p}_1 - \vec{p}_2)\vec{r} \equiv (p_1 - p_2)x \tag{6.12}$$

For simplicity further we will use the following designations:

$$p_{1,2} = \left|\vec{p}_{1,2}\right| = \left|\vec{p}_{\nu_{1,2}}\right|, \tag{6.13}$$

$$m_{1,2} = m_{\nu_{1,2}}, \tag{6.14}$$

$$E_{1,2} = E_{\nu_{1,2}} \tag{6.15}$$

We can write that

$$\left|\vec{p}_{1,2}\right|^2 = E_{1,2}^2 - m_{\nu_{1,2}}^2 \tag{6.16}$$

Using (6.16) and (6.11) we will get

$$|\vec{p}_1|^2 - |\vec{p}_2|^2 = E_1^2 - E_2^2 - m_1^2 + m_2^2 = (E_1 + E_2)\frac{m_1^2 - m_2^2}{2E_\pi} - m_1^2 + m_2^2$$
$$= \frac{m_1^2 - m_2^2}{2E_\pi}(E_1 + E_2 - 2E_\pi) \tag{6.17}$$

Finally, the phase difference is

$$\varphi(x,t) = (E_1 - E_2)t - (p_1 - p_2)x$$
$$= \frac{m_1^2 - m_2^2}{2E_\pi}t + \frac{m_1^2 - m_2^2}{E_1 + E_2}t - \frac{m_1^2 - m_2^2}{E_1 + E_2}t - (p_1 - p_2)x$$
$$= \frac{m_1^2 - m_2^2}{E_1 + E_2}t + \frac{m_1^2 - m_2^2}{2E_\pi(E_1 + E_2)}(E_1 + E_2 - 2E_\pi)t - (p_1 - p_2)x \Rightarrow$$

$$\varphi(x,t) = \frac{m_1^2 - m_2^2}{E_1 + E_2} t + (p_1 - p_2)\left[\frac{p_1 + p_2}{E_1 + E_2} t - x\right] \tag{6.18}$$

The oscillation length

Let's introduce the so-called oscillation length. If we have two flavor neutrinos $\begin{pmatrix}\nu_\mu \\ \nu_e\end{pmatrix}$ which are a superposition of mass states $\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix}$ than we can determine the probability of electron neutrino to be observed as another flavor muon neutrino is

$$P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\theta \sin^2 \frac{\Delta m^2}{4E} x, \tag{6.19}$$

where $\frac{\Delta m^2}{4E} = \frac{\pi}{L}$. So the oscillation length is

$$L = L_{12} = \frac{4\pi E}{\Delta m^2} \tag{6.20}$$

First of all let's discuss a physical meaning of the second term in (6.18). It can be neglected for the case of relativistic neutrinos ν_1, ν_2. We can write that

$$x = v_0 t = \frac{p_1 + p_2}{E_1 + E_2} t, \tag{6.21}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \gamma = \frac{p_0}{m} \Rightarrow \tag{6.22}$$

$$\beta^2 = \frac{p_0^2 - m^2}{p_0^2} = \frac{|\vec{p}|^2}{p_0^2} \tag{6.23}$$

Therefore the phase difference for relativistic neutrinos approximately equals

$$\varphi(x,t) \approx \frac{\Delta m^2}{2(E_1 + E_2)} t \tag{6.24}$$

Now we can calculate the uncertainty in the phase difference. It depends on the uncertainty in neutrino's energy which is proportional to the uncertainty in momentum

$$\delta E \sim \delta p \tag{6.25}$$

We can write that

$$\delta\varphi = \delta\left(\frac{\Delta m^2}{2E}\right) t = \frac{\Delta m^2}{2} \delta\left(\frac{1}{p}\right) t, \tag{6.26}$$

where

$$\delta\left(\frac{1}{p}\right) \sim \frac{1}{p^2} \delta p \tag{6.27}$$

Due to the Heisenberg uncertainty principle we know that

$$\delta E \sim \delta p \sim \frac{1}{\delta x}, \tag{6.28}$$

$\delta x -$ the uncertainty of the point where neutrino was produced in the source. Finally, the uncertainty in the phase difference is

$$\delta\varphi = \delta\left(\frac{\Delta m^2}{2E}\right)t = \frac{\Delta m^2}{2}\delta\left(\frac{1}{E}\right)x = \frac{\Delta m^2}{2}\delta\left(\frac{1}{p}\right)x = \frac{\Delta m^2}{2p^2}\delta p\, x \qquad (6.29)$$

The coherence length

Now we can introduce so-called coherence length in neutrino oscillations. The distance that neutrino travels from the source to the detector equals to the coherence length when the uncertainty in phase difference is rather small:

$$\delta\varphi \le 1(2\pi) \qquad (6.30)$$

So the coherence length is

$$L_{coh} = \frac{2p^2}{\delta p \Delta m^2}\delta\varphi|_{\delta\varphi=2\pi} = \frac{4\pi p^2}{\delta p \Delta m^2} \qquad (6.31)$$

The number of oscillation peaks

As soon as we introduced the coherent length we can now calculate very interesting characteristic of neutrino oscillation phenomenon that is the number of oscillation peaks that can be observed:

$$N_{max} = \frac{L_{coh}}{L_{12}} = \frac{2p^2(2\pi)}{\Delta m^2 \delta p}\frac{\Delta m^2}{4\pi p} = \frac{p}{\delta p} \qquad (6.32)$$

Normally when oscillations are observed $N_{max} \gg 1$ and the coherence condition is fulfilled.

The incapability of experiments on oscillations and double peak

Now we can discuss very interesting phenomenon related to the constraints on the possibility to observe oscillations due to uncertainties in the region where neutrinos were produced. If you measured very carefully the energy and momentum of initial particles π^+ and μ from the energy momentum conservation you can derive in which particular mass state neutrino appeared. Obviously once neutrino appears at the mass state there are no oscillations. So if neutrino is produced as ν_1 or ν_2 you will see two peaks in the energy distribution but you cannot observe oscillations.

a) If you measure the energy and momentum of initial particles p_π, p_μ with high accuracy than from the energy momentum conservation you can derive in which particular mass state ν_1 or ν_2 neutrino appears.
b) From the uncertainty principle we know that

$$\Delta p_\pi \Delta x_\pi \ge 1 \qquad (6.33)$$

It means that if Δp_π is very small than Δx_π is rather high.

c) The kinematic determination of the emitted neutrino is based on the well-known relation between the energy and the momentum:

$$m_\nu^2 = E_\nu^2 - p_\nu^2 \tag{6.34}$$

In order to distinguish neutrino mass state we have to consider the arrow in the determination of neutrino mass:

$$\Delta(m_\nu^2) = \sqrt{(2E_\nu)^2(\Delta E_\nu)^2 + (2p_\nu)^2(\Delta p_\nu)^2} \tag{6.35}$$

d) The neutrinos ν_1 and ν_2 can be distinguished only if the following condition is valid:

$$\Delta(m_\nu)^2 < |m_1^2 - m_2^2| \tag{6.36}$$

e) Further we will be interested only in second part under the square root in (6.35). So the condition to distinguish the mass states is

$$\sqrt{(2p_\nu)^2(\Delta p_\nu)^2} < |m_1^2 - m_2^2| \tag{6.37}$$

Therefore

$$\Delta p_\nu < \frac{\Delta m^2}{2p_\nu} \tag{6.38}$$

f) The uncertainty in momentum is inverse proportional to the uncertainty in the coordinate:

$$\Delta p_\nu \sim \frac{1}{\Delta x_\nu} \tag{6.39}$$

g) The oscillation length is

$$L_{12} \sim \frac{2p_\nu}{\Delta m^2} \tag{6.40}$$

Finally we can write that the uncertainty in the point where neutrino was produced is bigger than the oscillation length:

$$\Delta x > L_{12} \tag{6.41}$$

We see the violation of the coherence condition. It means that there is no possibility to observe the oscillation pattern.

Neutrino mixing and oscillations in matter

We are ready to finish our discussion of neutrino mixing and oscillations in vacuum. The next step will be considering neutrino oscillations in matter. We know that when a photon propagates in vacuum it moves with the speed of light and

$$E = |\vec{p}| \tag{6.42}$$

because it's mass is zero $m_\gamma = 0$. When the photon propagates in matter the energy momentum relation is violated and the mass becomes not zero $m_\gamma \neq 0$. The energy momentum relation will be changed to

$$E^2 = m_\gamma^2 + |\vec{p}|^2 \tag{6.43}$$

Obviously the same will happen with neutrino particle. Let's consider again two types of neutrinos ν_e, ν_μ and also two mass states ν_1, ν_2 with masses m_1, m_2. We would like to see what happens with the energy momentum relation due to interaction of neutrino with the environment. So we have to calculate the shift of the energy flavor neutrinos due to their interaction with the particles of the environment. Then we will add this shift to the effective Hamiltonian of neutrino propagation.

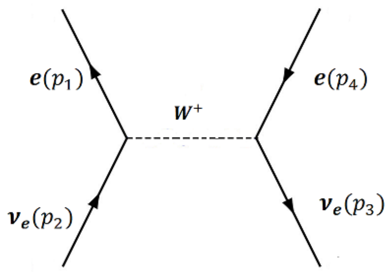
We suppose that matter is composed of neutrons, protons and electrons n, p, e. Also we suppose the neutrality, it means that the number of electrons equals to the number of protons. For the number densities of the corresponding particles we can write

$$n_e = n_p \tag{6.44}$$

Now we should consider the evolution of two flavor states $\begin{pmatrix}\nu_\mu \\ \nu_e\end{pmatrix}$ in presence of this particular matter. We should address to the Standard Model Lagrangian that describes an interaction of flavor neutrinos with other particles. We can classify in this Lagrangian the so-called contribution from charge current interaction and neutral current interaction. The charge current interaction is mediated by the charged bozon and the neutral current interaction – by neutral bozon.

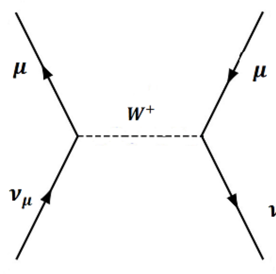
a) *Charge Current Interaction (CC)*

Let's write the typical Feinman diagram for our process (pic. 6.1).



Picture 6.1. Feinman diagram for electron neutrino interaction with electron.

According to the universality of weak interaction we can write the same diagram for muon neutrino (pic. 6.2). But in normal matter the number of muons is zero, it means that only first contribution survives.



Picture 6.2. Feinman diagram for neutrino interaction with muon.

From the Standard Model Lagrangian we can write the following expression that describes the interaction of electron neutrino with electrons:

$$\left(\frac{ig}{\sqrt{2}}\right)^2 \{\bar{e}_L(p_1)\gamma^\lambda \nu_{e_L}(p_2)\} \frac{-ig_{\lambda\rho}}{(p_2 - p_1)^2 - m_W^2} \{\bar{\nu}_{e_L}(p_3)\gamma_\rho e_L(p_4)\} \quad (6.45)$$

Lecture 7. Neutrino mixing and oscillations in matter

For understanding the nature of neutrino mixing and oscillations the relation between energy and momentum of neutrino turns out to be very important. This relation is sometimes called the dispersion relation $E = E(p)$. The next step is to discuss the characteristics of matter in which neutrino oscillations take place. We know that when a photon propagates in vacuum the relation between energy and momentum is very simple:

$$p_0^\gamma = |\vec{p}_\gamma| \tag{7.1}$$

The mass of the photon in vacuum is zero $m_\gamma = 0$. When a photon propagates in matter the mass becomes not zero $m_\gamma \neq 0$ and the energy momentum relation will be changed to

$$p_0^\gamma = \sqrt{m_\gamma^2 + |\vec{p}_\gamma|^2} \tag{7.2}$$

Taking into account this knowledge we can expect that in case of neutrino propagating in matter the relation between the energy and momentum will be violated too. We can also predict suppose that the presence of matter will change the neutrino mixing and oscillation pattern.

Neutrino mixing and oscillations in vacuum

We consider two mass states or physical neutrinos $\nu^p = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$ with the masses m_1, m_2. There are also flavor neutrinos ν^f that participate in the interaction Lagrangian. They were produced in the source and then detected. We postulate that flavor neutrinos are superpositions of the neutrino mass states:

$$\nu^f = U\nu^p \tag{7.3}$$

The mixing matrix can be parameterized by one quantity of the mixing angle:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{7.4}$$

The evolution equation for the physical neutrino states:

$$i\frac{d}{dt}\nu^p = H\nu^p, \tag{7.5}$$

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \tag{7.6}$$

Considering the case that all neutrinos are relativistic

$$E_\alpha = \sqrt{m_\alpha^2 + |\vec{p}_\alpha|^2} \cong |\vec{p}_\alpha|^2 + \frac{m_\alpha^2}{2|\vec{p}_\alpha|}, \alpha = 1,2 \tag{7.7}$$

So for the evolution Hamiltonian we get the following expression:

$$H = \left(|\vec{p}| + \frac{m_1^2 + m_2^2}{4|\vec{p}|}\right)\hat{I} - \frac{\Delta m^2}{4|\vec{p}|}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.8}$$

$\hat{I}$ − unix matrix. According to neutrino mixing

$$\nu^p = U^+\nu^f \tag{7.9}$$

We put (7.9) in (7.5), substituting the evolution in time to the evolution in space

$$i\frac{d}{dx}\nu^f = UHU^+\nu^f = H'_{vac}\nu^f$$
$$= \left(|\vec{p}| + \frac{m_1^2 + m_2^2}{4|\vec{p}|}\right)\hat{I} - \frac{\Delta m^2}{4|\vec{p}|}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \tag{7.10}$$

$UHU^+ = H'_{vac}$ − the Hamiltonian prime in vacuum. We have obtained the probability of neutrino oscillations between electron muon flavors in the following way

$$P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\theta \sin^2 \frac{\Delta m^2}{4E}x, E = |\vec{p}| \tag{7.11}$$

If we know the elements of the Hamiltonian H'_{vac} for any type of neutrino oscillations in vacuum:

$$H'_{vac} = \begin{pmatrix} H_{ii} & H_{ij} \\ H_{ji} & H_{jj} \end{pmatrix} \tag{7.12}$$

then the amplitude of oscillations in (7.11) can be expressed as

$$\sin^2 2\theta = \frac{(H_{ij} + H_{ji})}{(H_{jj} + H_{ii})^2 + (H_{ij} + H_{ji})^2}, \tag{7.13}$$

and the other multiplier as

$$\sin^2 \frac{\Delta m^2}{4E}x = \sin^2 \frac{2\pi x}{L}, \tag{7.14}$$

$$L = \frac{2\pi}{\left[(2H_{ij})^2 + (H_{jj} + H_{ii})^2\right]^{3/2}} \tag{7.15}$$

Neutrino ν_e and ν_μ oscillations in matter

Let's return to our main problem that was announced and deal with the particular case of two neutrinos ν_e and ν_μ propagating in matter. We suppose that matter is composed of neutrons, protons and electrons n, p, e. In addition we suppose the matter is neutral, it means that the number of electrons in a cm^3 equals to the number of protons. For the number densities of the corresponding particles we can write that

$$n_e = n_p \tag{7.16}$$

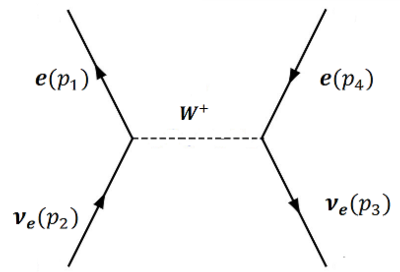
Such "normal" matter presents, for example, on the Earth and on the Sun. Now we have to specify the consequence of neutrino interaction with matter. According to the analogy with a photon propagating in vacuum and in matter we can expect that these interactions will change the effective energy of neutrino in matter in respect to what we have in vacuum. So we suppose that there will be the violation of relation between energy and momentum and some additional term so-called effective or potential energy will appear.

Interaction of electron neutrino with electrons

We have to start with the Standard Model Lagrangian that describes an interaction of flavor neutrinos first of all with electrons. The interaction of neutrino with the set of particles n, p, e can be classified on two different types. The first type is called the interaction of charged currents, the second type – the interaction of neutral currents.

Charged currents interaction (contribution to the effective energy of ν_e and ν_μ in matter)

The typical Feinman diagram for electron neutrino ν_e (pic. 7.1).



Picture 7.1. Feinman diagram for interaction of electron neutrino with electron.

Addressing to our knowledge of the Standard Model Lagrangian let's write the mathematical expression that describes the interaction of electron neutrino with electrons:

$$\Delta\mathcal{L}_{\nu_e}^{CC} = \left(\frac{ig}{\sqrt{2}}\right)^2 \{\bar{e}_L(p_1)\gamma^\lambda \nu_{e_L}(p_2)\} \frac{-ig_{\lambda\rho}}{(p_2-p_1)^2-m_W^2} \{\bar{\nu}_{e_L}(p_3)\gamma_\rho e_L(p_4)\}, \tag{7.17}$$

$$\bar{e}_L = \frac{1}{2}(1-\gamma_5)e^{T*}\gamma_0, \tag{7.18}$$

where $\frac{-ig_{\lambda\rho}}{(p_2-p_1)^2-m_W^2}$ – the Green function of W-bozon, but we also exclude a second term that is suppressed in case when the energy is smaller than the mass of W-bozon. If we also neglect the momentums $p_{1,2}$ in respect to the mass of W-bozon m_W we can write that

$$\Delta\mathcal{L}_{\nu_e}^{CC} \cong -\frac{4G_F}{\sqrt{2}} \{\bar{e}_L(p_1)\gamma^\lambda \nu_{e_L}(p_2)\}\{\bar{\nu}_{e_L}(p_3)\gamma_\lambda e_L(p_4)\} \tag{7.19}$$

We have got the result of consideration these interactions as the local Fermi interactions where

$$G_F = \frac{g^2}{8m_W^2}\sqrt{2} \tag{7.20}$$

Now let's check that it is possible for this expression to change the positions of fields. It was shown by Feirz and called the Feirz transformation:

$$\Delta\mathcal{L}_{\nu_e}^{CC} = -\frac{4G_F}{\sqrt{2}}\{\bar{e}_L(p_1)\gamma^\lambda e_L(p_4)\}\{\bar{\nu}_{e_L}(p_3)\gamma_\lambda \nu_{e_L}(p_2)\} \tag{7.21}$$

The initial state of this particular process is composed of electron's and neutrino's fields. The final state is also composed of these two fields. We are interested on how this particular part of Lagrangian changes the effective energy of particular neutrino. We should use the vacuum electron wave functions and make an average. So we have derived an addition to the effective Lagrangian:

$$\langle e\nu|H_{\nu_e}^{CC}|e\nu\rangle = \bar{\nu}_e V_{\nu_e}\nu_e \tag{7.22}$$

After that we can write the addition to the Lagrangian for the charge current interaction of electron neutrino with electrons of the background:

$$\Delta\mathcal{L}_{\nu_e}^{CC} = -\sqrt{2}G_F n_e \bar{\nu}_{e_L}\gamma_0\nu_{e_L} \tag{7.23}$$

The number density of electrons is

$$\langle\bar{e}\gamma_0 e\rangle = n_e \tag{7.24}$$

The continuous relation in classical electrodynamics:

$$\frac{\partial\rho}{\partial t} + div\vec{j} = 0 \tag{7.25}$$

If we consider a quantum electrodynamics, in particular, the Dirac equation for a charged particle, for example, an electron in a magnetic field, there will be the following transformation:

$$p_\mu \to p_\mu + e_0 A_\mu, A^\mu = (\varphi, \vec{A}) \tag{7.26}$$

The Dirac equation:

$$\left(\gamma^\mu(p_\mu + e_0 A_\mu) - m\right)\Psi(x) = 0 \tag{7.27}$$

We can introduce the matrices:

$$\bar{\alpha} = \gamma^0\bar{\gamma}, \beta = \gamma^0 \tag{7.28}$$

Therefore

$$i\frac{\partial\Psi}{\partial t} = -i\bar{\alpha}\bar{\nabla}\Psi + e_0\bar{\alpha}\vec{A}\Psi + m\beta\Psi \tag{7.29}$$

The similar equation for Ψ^+:

$$-i\frac{\partial\Psi^+}{\partial t} = -i\bar{\nabla}\Psi^+\bar{\alpha} + e_0\Psi^+\bar{\alpha}\vec{A} + m\Psi^+\beta \tag{7.30}$$

Multiplying (7.29) by Ψ^+ and (7.30) by Ψ we get

$$\frac{\partial}{\partial t}(\Psi^+\Psi) + div(\Psi^+\bar{\alpha}\Psi) = 0 \tag{7.31}$$

Multiplying (7.30) by e_0, we get the following expressions for density and current:

$$\rho = e_0\Psi^+\Psi, \tag{7.32}$$

$$\vec{j} = e_0\Psi^+\bar{\alpha}\Psi \tag{7.33}$$

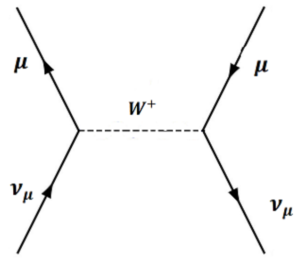
We suppose that the matter is at rest or nonrelativistic. I would like to say that many years ago I proposed that it's possible to account for the motion of matter. With my postgraduate students we generalized this approach to the case when matter is moving and had found very important changes in the neutrino mixing and oscillation pattern. Such relativistic matter can be found in astrophysics.

Interaction of muon neutrino with electrons

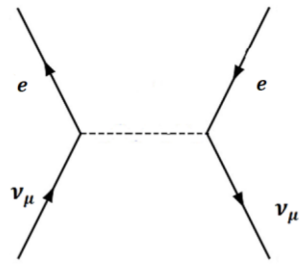
Let's consider charged current interaction of muon neutrino with electrons. We can immediately say that

$$\Delta\mathcal{L}_{\nu_\mu}^{CC} = 0 \tag{7.34}$$

There is universality between electron and muon neutrino interactions: electron neutrino interacts with electrons and muon neutrino – with muons. But there are no muons in our environment $n_\mu = 0$. So in our matter the interaction as on pic.7.1 is not possible. There is no such diagram as pic.7.2 in the Standard Model because the flavor is a good quantum number which is conserved. But once we introduce the mixing between ν_e and ν_μ we violate the flavor conservation law.



Picture 7.1. Feinman diagram for muon neutrino interaction with muon.

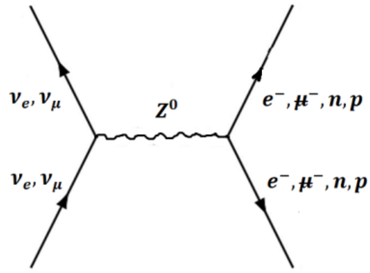


Picture 7.2. Feinman diagram for muon neutrino interaction with muon.

Neutral currents interaction

Let's turn to the case of neutral currents and consider the interaction due to emission and absorption of neutral bosons. Obviously the charged current interaction is mediated by charged boson. The Feinman diagram for neutral currents interactions shown in the pic.7.3.

There are no muons among the final particles in pic.7.3. Muons in reasonable amount can appear under huge temperature and huge density in some extreme environments of the early universe or in some catastrophic events in astrophysics.



Picture 7.3. Feinman diagram for neutral currents interactions.

The contribution of the neutral currents interaction in Lagrangian:

$$\Delta\mathcal{L}_\nu^{NC} = -\frac{4G_F}{\sqrt{2}}\left\{\bar{f}\gamma^\lambda\left(I_{3L}\frac{1-\gamma_5}{2}Q\delta u^2\theta_N\right)f\right\}\{\bar{\nu}_L\gamma_\lambda\nu_L\}, \quad (7.35)$$

$f = e, n, p -$ the particles of the environment. For f particles we can write the known values for the following charges:

	I_{3L}	Q
e	$^3/_2$	-1
p	$^1/_2$	$+1$
n	$-^1/_2$	0

Table. 7.1. Charged characteristics for the particles in the environment.

Therefore

$$\Delta\mathcal{L}_\nu^{NC} = -\sqrt{2}G_F\left[\sum_{f=e,n,p} n_f\left(I_{3L}^f - \sin^2\theta_N \cdot Q^f\right)\right]\bar{\nu}_L\gamma_0\nu_L \quad (7.36)$$

We again suppose that matter is not moving. We remember that the number of electrons and protons are the same: $n_e = n_p$. But the electric charges and the third components of isospin of electrons and protons are opposite. Finally, we can write

$$\Delta\mathcal{L}^{NC+CC} = \sum_{l=e,\mu}\bar{\nu}_{e_L}\gamma_0 V_{\nu_e}\nu_{e_L}, \quad (7.37)$$

where the effective potentials for electron and muon neutrino are

$$V_{\nu_e} = \sqrt{2}G_F\left(n_e - \frac{1}{2}n_n\right) = \frac{G_F}{\sqrt{2}}(2n_e - n_n) \quad (7.38)$$

$$V_{\nu_\mu} = \frac{1}{\sqrt{2}}G_F n_n \quad (7.39)$$

Accounting for the additional contribution to the Lagrangian due to charged currents and neutral currents interactions $\Delta\mathcal{L}^{NC+CC}$ we can modify the vacuum Dirac equation for the neutrino wave function:

$$(p - m)\Psi(x) = 0 \tag{7.40}$$

to

$$(\gamma_0 p_0 - \vec{\gamma}\vec{p} - m)\Psi = \gamma_0 V\Psi \Rightarrow \tag{7.41}$$

$$\gamma_0(p_0 - V) = \vec{\gamma}\vec{p} + m \Rightarrow \tag{7.42}$$

$$p_0 = \sqrt{m^2 + |\vec{p}|^2} + V \tag{7.43}$$

It looks like neutrino gets an additional potential energy.

Mixing and oscillations in matter

Further we will use the designation accounting for the effect of matter: $a \to \tilde{a}$. Let's recall once again the equation for flavor neutrino evolution in vacuum:

$$i\frac{d}{dx}\nu^f = H'\nu^f \tag{7.44}$$

The same equation in matter:

$$i\frac{d}{dx}\nu^f = \widetilde{H}\nu^f, \tag{7.45}$$

$\widetilde{H} -$ the modified evolution Hamiltonian. The flavor state is determined by the Standard Model and it's definition isn't violated by presence or not presence of matter. The Hamiltonian in matter equals to

$$\widetilde{H} = H' + \begin{pmatrix} \sqrt{2}G_F\left(n_e - \frac{1}{2}n_n\right) & 0 \\ 0 & -\frac{1}{\sqrt{2}}G_F n_n \end{pmatrix} \Rightarrow \tag{7.46}$$

$$\widetilde{H} = \left(E + \frac{m_1^2 + m_2^2}{4E}\frac{1}{\sqrt{2}}G_F n_n\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \tag{7.47}$$

We notice that only charged currents interaction of electron neutrino with electrons contribute to the shift of the oscillation probability in matter. Sometimes the second part in (7.47) is called $\frac{1}{2E}\widetilde{M}^2$, where

$$\widetilde{M}^2 = \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos 2\theta + 2A & \Delta m^2 \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta m^2 \cos 2\theta \end{pmatrix}, \tag{7.48}$$

$$A = 2\sqrt{2}G_F n_e E \tag{7.49}$$

Once we have derived the evolution Hamiltonian accounting for matter effect

$$\widetilde{H} = \begin{pmatrix} \widetilde{H}_{11} & \widetilde{H}_{12} \\ \widetilde{H}_{21} & \widetilde{H}_{22} \end{pmatrix}, \tag{7.50}$$

the probability of oscillations of electron neutrino to muon neutrino in matter composed of $n_e = n_p, n_n$:

$$P_{\nu_e \to \nu_\mu}(x) = \sin^2 2\tilde{\theta} \sin^2 \frac{\pi x}{L_{matt}} \tag{7.51}$$

So the mixing angle and the oscillation length were modified. The amplitude of oscillations is

$$\sin^2 2\tilde{\theta} = \frac{(\Delta m^2 \sin 2\theta)^2}{(\Delta m^2 \cos 2\theta - A)^2 + (\Delta m^2 \sin 2\theta)^2} \tag{7.52}$$

If we put $n_e = 0, A = 0$ then $\tilde{\theta} = \theta$. The oscillation length in matter also depends on density of electrons:

$$L_{matt} = \frac{2\pi}{\sqrt{\left(\frac{\Delta m^2}{2E}\cos 2\theta - \sqrt{2} G_F n_e\right)^2 + \left(\frac{\Delta m^2}{2E}\sin 2\theta\right)^2}} \tag{7.53}$$

Mikheev-Smirnov-Wolfenstien effect

There is the resonance in neutrino oscillations $\nu_e \to \nu_\mu$ when

$$\Delta m^2 \cos 2\theta = A \tag{7.54}$$

then the amplitude of oscillations

$$\sin^2 2\tilde{\theta} = 1 \tag{7.55}$$

This phenomenon was first predicted and observed theoretically by two gifted Soviet Union scientists Mikheev and Smirnov in 1985. Firstly it was called the Mikheev-Smirnov effect but just very soon this name was corrected because Hans Betta have wrote a paper where he recall that Mikheev and Smirnov used the same calculations that were addressed to an early paper of american scientist Wolfenstien (1978). So now the resonance in flavor neutrino oscillations is called Mikheev-Smirnov-Wolfenstien effect. This effect is used to explain the observed flux of neutrinos from the Sun. The present solution of solar neutrino problem due to this effect is called the Large Mixing Angle Mikheev-Smirnov-Wolfenstien Solution of Solar neutrino problem.

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