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teach-in
ЛЕКЦИИ УЧЕНЫХ МГУ

МАТЕМАТИЧЕСКИЙ АНАЛИЗ. ИЗБРАННЫЕ ГЛАВЫ. ЧАСТЬ 4

СОЛОДОВ
АЛЕКСЕЙ ПЕТРОВИЧ

МЕХМАТ МГУ

БЛАГОДАРИМ ЗА ПОДГОТОВКУ КОНСПЕКТА
СТУДЕНТКУ ФИЗИЧЕСКОГО ФАКУЛЬТЕТА МГУ
ЧУГРЕЕВУ ГАЛИНУ НИКОЛАЕВНУ

Содержание

Семинар 1. Вычисление интегралов с помощью специальных функций (часть I).

Гамма-функция.

$$\Gamma(x) = \int\limits_0^{\infty} t^{x-1} e^{-t} dt\,, x > 0$$

Свойства:

1. $\Gamma \in C^{\infty}(R_+)$
2. $\Gamma(1) = 1$
3. $\Gamma(x+1) = x\ \Gamma(x)$
4. $\Gamma(n+1) = n!$
5. $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, $x \in (0;\ 1) \Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Бета-функция.

$$B(x,y) = \int\limits_0^{1} t^{x-1}(1-t)^{y-1} dt\,, x, y > 0$$

Свойства:

1. $B \in C^{\infty}(R_+ \times R_+)$
2. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
3. Если , $x + y \in Z$, то $B(x,y)$ выражается через элементарные функции.

№1. $\int_a^b (x-a)^m (b-x)^n\,dx,\ 0 < a < b$

Замена: $x = a + (b-a)t$

$$\int_a^b (x-a)^m (b-x)^n\,dx = \int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a)dt$$

$$= (b-a)^{m+n+1} B(m+1, n+1), \qquad m+n \in Z$$

№2. $\int_0^\infty \frac{x^m}{(a+bx^n)^p}\,dx,\ a, b, n > 0$

$$\int_0^\infty \frac{x^m}{\left(a+bx^n\right)^p}dx = \left| a+bx^n = \frac{a}{t}, \qquad x = \left(\frac{a}{b}\frac{1-t}{t}\right)^{\frac{1}{n}} \right| =$$

$$\frac{1}{n}\int_0^1 \left(\frac{a}{b}\frac{1-t}{t}\right)^{\frac{m}{n}} \frac{t^p}{a^p} \left(\frac{a}{b}\right)^{\frac{1}{n}} \left(\frac{1-t}{t}\right)^{\frac{1}{n}-1} \frac{dt}{t^2} =$$

$$\frac{1}{n}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} a^{-p} \int_0^1 t^{1-\frac{1}{n}-\frac{m}{n}+p-2} (1-t)^{\frac{m+1}{n}-1} dt = \frac{1}{n}\left(\frac{a}{b}\right)^{\frac{m+1}{n}} a^{-p} B\left(\frac{m+1}{n}, p - \frac{m+1}{n}\right)$$

№3. $\int_0^{\frac{\pi}{2}} (\sin x)^m x\,(\cos x)^n\,dx = \int_0^{\frac{\pi}{2}} (\sin x)^m x\,(\cos x)^{n-1} d\sin x =$

$$\frac{1}{2}\int_0^{\pi/2} (\sin x)^{m-1} x\,(\cos x)^{n-1} d(\sin x)^2 = \left| (\sin x)^2 = t \right|$$

$$= \frac{1}{2}\int_0^1 t^{\frac{m-1}{2}} (1-t)^{\frac{n-1}{2}} dt = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Чтобы этот интеграл сходился, нужно, чтобы $m, n > 1$, а чтобы выражался через элементарные функции, нужно, чтобы $m+n$ была четной.

Частные случаи:

1. $m+n=0\ \int_0^{\frac{\pi}{2}}(tg\,x)^m dx = \frac{1}{2}\,B\left(\frac{m+1}{2},\frac{1-m}{2}\right) = \frac{1}{2}\,\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1-m}{2}\right)}{\Gamma(1)} =$

$$= \frac{\pi}{2\ \sin\frac{\pi(m+1)}{2}} = \frac{\pi}{2\ \cos\frac{\pi m}{2}}$$

2. $\int_0^{\frac{\pi}{2}}\sqrt{tg\,x}\,dx = \frac{\pi}{\sqrt{2}}$

№4. $\int_0^{\infty}\frac{(x-a)^m(b-x)^n}{(x+c)^{m+n+2}}dx\,, 0<a<\ b, c>0$

$$\int_0^{\infty}\frac{(x-a)^m(b-x)^n}{(x+c)^{m+n+2}}dx = \left|\begin{array}{c} t=\frac{x-a}{b-x} \\ x=\frac{a+bt}{1+t} \\ x-a=\frac{(b-a)t}{1+t} \\ b-x=\frac{b-a}{1+t} \\ dx=\frac{b-a}{(1+t)^2} \end{array}\right| = \int_0^{\infty}\frac{(b-a)^{m+n+1}t^m}{(1+t)^{m+n+2}\ \frac{(a+c+(b+c)t)^{m+n+2}}{(1+t)^{m+n+2}}}dt =$$

$$= \left(\frac{a+c}{b+c}\right)^{m+1}(b-a)^{m+n+1}(a+c)^{-m-n-2}\,B(m+1,n+1)$$

№5. $\int_0^{\pi}\frac{(\sin x)^{n-1}x}{(1+k\cos x)^n}dx = \int_0^{\pi}\frac{(\sin x)^{n-2}x}{(1+k\cos x)^n}d\,cos\,x =$

$$= \int_0^{\pi}\frac{(1-\cos x)^{\frac{n}{2}-2}(1+\cos x)^{\frac{n}{2}-2}}{(1+k\cos x)^{\frac{n}{2}-1+\frac{n}{2}-1+2}}d\,cos\,x = |-\cos x = t| =$$

$$= \int_{-1}^{1} \frac{(1-t)^{\frac{n}{2}-2}(1+t)^{\frac{n}{2}-2}}{(1-kt)^{\frac{n}{2}-1+\frac{n}{2}-1+2}} d\,t$$

Семинар 2. Вычисление интегралов с помощью специальных функций (часть II).

Важные результаты прошлого семинара:

$$\int_0^\infty \frac{x^m}{(1+x^n)^p}\,dx = \frac{1}{n}\,B\left(\frac{m+1}{n}, p-\frac{m+1}{n}\right)$$

$$\int_0^{\frac{\pi}{2}} (\sin x)^m x\,(\cos x)^n\,dx = \frac{1}{2}\,B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

№1. $\int_0^\infty \frac{arctg\,x}{x^m}dx = |1 < m < 2\,| = \frac{-1}{m-1}\int_0^\infty arctg\,x dx^{1-m} =$

$\frac{1}{m-1}\int_0^\infty x^{1-m}\,d\,arctg\,x = \frac{1}{m-1}\int_0^\infty \frac{x^{1-m}}{1+x^2}dx = \frac{1}{2(m-1)}\,B\left(\frac{m}{2}, 1-\frac{m}{2}\right) = \frac{\pi}{2(m-1)\sin\frac{\pi m}{2}}$

№2. $\int_0^\infty \frac{\ln(1+x)}{x^m}dx = |1 < m < 2\,| = \frac{-1}{m-1}\int_0^\infty \ln(1+x)\,dx^{1-m} =$

$\frac{1}{m-1}\int_0^\infty x^{1-m}\,d\,\ln(1+x) = \frac{1}{m-1}\int_0^\infty \frac{x^{1-m}}{1+x^2}dx = \frac{1}{m-1}\,B(2-m,\ m-1) = \frac{\pi}{(m-1)\sin\frac{\pi(m-1)}{2}}$

№3. $\int_0^\infty \frac{e^{\alpha x}}{1+e^x}dx = |0 < \alpha < 1\,| = \begin{vmatrix} e^x = t \\ x = \ln t \\ dx = \frac{dt}{t}\end{vmatrix} = \int_0^\infty \frac{t^{\alpha-1}}{1+t}dt = B(\alpha, 1-\alpha) = \frac{\pi}{\sin \pi\alpha}$

№3. $\int_{-\infty}^{\infty}\frac{ch\,\alpha x}{ch\,\beta x}dx = |0<\alpha<\beta\,| = \int_{-\infty}^{\infty}\frac{e^{\alpha x}+e^{-\alpha x}}{e^{\beta x}+e^{-\beta x}}dx = \int_{-\infty}^{\infty}\frac{e^{\alpha x}}{e^{\beta x}+e^{-\beta x}}dx +$

$$\int_{-\infty}^{\infty}\frac{e^{-\alpha x}}{e^{\beta x}+e^{-\beta x}}dx = \int_{-\infty}^{\infty}\frac{e^{(\alpha+\beta)x}}{e^{2\beta x}+1}dx + \int_{-\infty}^{\infty}\frac{e^{(\beta-\alpha)x}}{e^{2\beta x}+1}dx = |\,2\beta x = t|$$

$$= \frac{1}{2\beta}\int_{-\infty}^{\infty}\frac{e^{\frac{\alpha+\beta}{2\beta}x}}{e^{t}\,+1}dt + \frac{1}{2\beta}\int_{-\infty}^{\infty}\frac{e^{\frac{\beta-\alpha}{2\beta}x}}{e^{t}\,+1}dt = \frac{\pi}{2\beta\sin\pi\dfrac{\alpha+\beta}{2\beta}} + \frac{\pi}{2\beta\sin\pi\dfrac{\beta-\alpha}{2\beta}}$$

$$= \frac{\pi}{\beta\cos\dfrac{\pi\alpha}{2\beta}}$$

№4. $\int_{-\infty}^{\infty}\frac{sh\,\alpha x}{sh\,\beta x}dx = |0<\alpha<\beta\,| = \int_{-\infty}^{\infty}\frac{e^{\alpha x}-e^{-\alpha x}}{e^{\beta x}-e^{-\beta x}}dx$

№5. $\int_{-\infty}^{\infty}\frac{e^{\alpha x}}{1-e^{x}+e^{2x}}dx = |0<\alpha<2\,| = \int_{-\infty}^{\infty}\frac{e^{\alpha x}+e^{(\alpha+1)x}}{1+e^{3x}}dx = |\,3x=t\,| =$

$\frac{1}{3}\int_{-\infty}^{\infty}\frac{e^{\frac{\alpha}{3}t}}{1+e^{t}}dt + \frac{1}{3}\int_{-\infty}^{\infty}\frac{e^{\frac{\alpha+1}{3}t}}{1+e^{t}}dt = \frac{\pi}{3\sin\pi\frac{\alpha}{3}} + \frac{\pi}{2\beta\sin\pi\frac{(\alpha+1)}{3}}$

№6. $\int_{0}^{\infty}\frac{x^{\alpha}}{(x+1)(x+2)}dx = |-1<\alpha<0\,| = \int_{0}^{\infty}x^{\alpha}\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx =$

$$= \int_{0}^{\infty}\frac{x^{\alpha}}{x+1}dx - \int_{0}^{\infty}\frac{x^{\alpha}}{x+2}dx = |\,x=2t\,| = B(\alpha+1,-\alpha) - 2^{\alpha}\,B(\alpha+1,-\alpha)$$

$$= (1-2^{\alpha}\,)\,\frac{\pi}{\sin(-\pi\alpha)}$$

$$\int_{0}^{\infty}\frac{x^{\alpha}}{(x+1)(x+2)}dx = |0<\alpha<1\,| = \int_{0}^{\infty}x^{\alpha-1}\frac{x}{(x+1)(x+2)}dx$$

$$= \int_{0}^{\infty}x^{\alpha-1}\left(\frac{2}{x+2}-\frac{1}{x+1}\right)dx = 2\int_{0}^{\infty}\frac{x^{\alpha-1}}{x+2}dx - \int_{0}^{\infty}\frac{x^{\alpha-1}}{x+1}dx$$

$$= 2^{\alpha}\,B(\alpha,1\,-\alpha) - B(\alpha,1-\alpha) = (2^{\alpha}\,-1)\,\frac{\pi}{\sin\pi\alpha}$$

№7. $\int_0^\infty \frac{(1-x)^\alpha (1+x)^\alpha}{1+x^2} dx = \left| \begin{matrix} \frac{1+x}{1-x} = t \\ 1+x = t - tx \\ x = \frac{t-1}{t+1} \end{matrix} \right| = \int_0^\infty \frac{\frac{2^\alpha}{(1+t)^\alpha} - \frac{2^{1-\alpha}}{(1+t)^{1-\alpha}}}{1+\frac{(t-1)^2}{(1+t)^2}} \frac{2\ \ dt}{(1+t)^2} =$

$\int_0^\infty \frac{t^{1-\alpha}\, dt}{(1+t)(1+t^2\ )}$

$$1 < \alpha < 2, \qquad \left| \frac{1}{(1+t)\left(1+t^2\ \right)} = \frac{1}{2}\left(\frac{1}{1+t} - \frac{t-1}{1+t^2}\ \right) \right|$$

$$= \int_0^\infty \frac{t^{1-\alpha}\, dt}{1+t} - \int_0^\infty \frac{t^{2-\alpha}\, dt}{1+t^2} + \int_0^\infty \frac{t^{1-\alpha}\, dt}{1+t^2} =$$

$$= B(2-\alpha, \alpha-1) - \frac{1}{2} B\left(\frac{3-\alpha}{2}, 1 - \frac{3-\alpha}{2}\right) + \frac{1}{2} B\left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right)$$

$$= \frac{\pi}{\sin \pi(\alpha-1)} - \frac{1}{2} \frac{\pi}{\sin \frac{\pi(3-\alpha)}{2}} + \frac{\pi}{\sin \frac{\pi\alpha}{2}} = \frac{\pi}{2 \sin \frac{\pi\alpha}{2}} + \frac{\pi}{2 \cos \frac{\pi\alpha}{2}} - \frac{\pi}{\sin \pi\alpha}$$

$$= \frac{\pi(\sin \frac{\pi\alpha}{2} + \cos \frac{\pi\alpha}{2} - 1)}{\sin \pi\alpha}$$

$$0 < \alpha < 1, \qquad \left| \frac{t}{(1+t)\left(1+t^2\ \right)} = \frac{1}{2}\left(\frac{-1}{1+t} + \frac{t+1}{1+t^2}\ \right) \right|$$

$$= -\int_0^\infty \frac{t^{-\alpha}\, dt}{1+t} + \int_0^\infty \frac{t^{1-\alpha}\, dt}{1+t^2} + \int_0^\infty \frac{t^{-\alpha}\, dt}{1+t^2} =$$

$$= -B(1-\alpha, \alpha) + \frac{1}{2} B\left(1 - \frac{\alpha}{2}, \frac{\alpha}{2}\right) + \frac{1}{2} B\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right) =$$

$$= \frac{-\pi}{\sin \pi\alpha} + \frac{1}{2}\frac{\pi}{\sin \frac{\pi(1+\alpha)}{2}} + \frac{\pi}{2\sin \frac{\pi\alpha}{2}} = \frac{\pi(\sin \frac{\pi\alpha}{2} + \cos \frac{\pi\alpha}{2} - 1)}{\sin \pi\alpha}$$

$$-1 < \alpha < 0, \qquad \left|\frac{t^2}{(1+t)\left(1+t^2\right)} = \frac{1}{2}\left(\frac{1}{1+t} + \frac{t-1}{1+t^2}\right)\right|$$

$$= \int_0^\infty \frac{t^{1-\alpha}\,dt}{1+t} + \int_0^\infty \frac{t^{-\alpha}\,dt}{1+t^2} - \int_0^\infty \frac{t^{-\alpha}\,dt}{1+t^2} =$$

$$= B(-\alpha, 1+\alpha) + \frac{1}{2}B\left(\frac{1-\alpha}{2}, \frac{1+\alpha}{2}\right) - \frac{1}{2}B\left(-\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right) =$$

$$= \frac{-\pi}{\sin \pi\alpha} + \frac{1}{2}\frac{\pi}{\sin \frac{\pi(1-\alpha)}{2}} + \frac{\pi}{2\sin \frac{\pi\alpha}{2}}$$

Семинар 3. Интегрирование по параметру

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt\,, x > 0, \qquad \Gamma(x) \in \Gamma^\infty(0, +\infty)$$

$$\Gamma'\ (x) = \int_0^\infty t^{x-1} e^{-t} \ln t\, dt\,, \qquad \Gamma''\,(x) = \int_0^\infty t^{x-1} e^{-t} ln^2\ t\ dt$$

$$\Gamma(x+1) = x\,\Gamma(x), \qquad \Gamma'\ (x+1) = \ \Gamma\ \ \ (x) + x\Gamma'\ (x),$$

$$\Gamma''\,(x+1) = 2\Gamma'\ (x) + x\Gamma''\,(x)$$

№1. $\int_0^\infty \frac{\ln x}{\sqrt{x}(x^2+1)}\ dx = F'\left(-\frac{1}{2}\right) = -\frac{\pi^2\sqrt{2}}{4}$

$$F(p) = \int_0^\infty \frac{x^p}{x^2\ +1}\ dx = \ \frac{1}{2}\ B\left(\frac{p+1}{2}, \frac{1-p}{2}\right) = \ \frac{1}{2} \frac{\pi}{\sin\frac{\pi(p+1)}{2}} = \frac{1}{2} \frac{\pi}{\cos\frac{\pi p}{2}},$$

$$F'(p) = \int_0^\infty \frac{x^p\ \ln x}{x^2\ +1}\ dx = \ \frac{\pi^2\ \sin\frac{\pi p}{2}}{4cos^2\ \frac{\pi p}{2}}, -1 < p < 1$$

$$\int_\infty^\infty \frac{x^m}{(x^n\ +1)^p}\ dx = \ \frac{1}{n}\ B(\frac{m+1}{n}, p - \frac{m+1}{n})$$

$$F'\left(-\frac{1}{2}\right) = -\ \frac{\pi^2\ \frac{\sqrt{2}}{2}}{4\,\frac{1}{2}} = -\frac{\pi^2\sqrt{2}}{4}$$

№2. $\int_{-1}^{1} \frac{\ln\frac{1+x}{1-x}}{\sqrt[3]{(1-x^2)(1+x)}}\, dx = F'\left(\frac{2}{3}\right) = \frac{\pi^2/2}{3/4} = \frac{2}{3}\pi^2$

$$F(p) = \int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^p \frac{dx}{1+x}, \qquad F'(p) = \int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^p \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{1+x},$$

$$F(p) = \int_{-1}^{1} (1+x)^{p-1}(1-x)^{-p}dx = |\, x = -1 + 2t|$$

$$= 2\int_{0}^{1} 2^{p-1}\, t^{p-1} 2^{-p}(1-t)^{-p}dt = B(p, 1-p) = \frac{\pi}{\sin \pi p}$$

$$F'(p) = -\frac{\pi^2 \cos \pi p}{sin^2\, \pi p}, \qquad F'\left(\frac{2}{3}\right) = \frac{\pi^2/2}{3/4} = \frac{2}{3}\pi^2$$

№3. $\int_{-\infty}^{\infty} \frac{x\, e^{x/2}}{e^{2x}+1}\, dx = F'\left(\frac{1}{2}\right) = -\frac{\pi^2\sqrt{2}}{4}$

$$F(p) = \int_{-\infty}^{\infty} \frac{e^{xp}}{e^{2x}+1}\, dx, \int_{0}^{\infty} \frac{e^{x\alpha}}{e^{x}+1}\, dx = \frac{\pi}{\sin \pi p}, F'(p) = \int_{-\infty}^{\infty} \frac{x\, e^{xp}}{e^{2x}+1}\, dx$$

$$F(p) = |2x = t| = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{pt/2}}{e^{t}+1}\, dt = \frac{\pi}{2\sin\frac{\pi p}{2}}$$

$$F'(p) = -\frac{\pi^2 \cos\frac{\pi p}{2}}{4\, sin^2\, \frac{\pi p}{2}}, \qquad F'\left(\frac{1}{2}\right) = -\frac{\pi^2\sqrt{2}}{4}$$

№3. $\int_{-\infty}^{\infty}\frac{x^2}{ch\,x}\,dx = F''(0) = \frac{\pi^3}{4}$

$$F(p) = \int_{-\infty}^{\infty}\frac{ch\,px}{ch\,x}\,dx\,, -1 < p < 1\,, F'(p) = \int_{-\infty}^{\infty}\frac{x\,sh\,px}{ch\,x}\,dx,$$

$$F''(p) = \int_{-\infty}^{\infty}\frac{x^2\;ch\,px}{ch\,x}\,dx$$

$$F(p) = \frac{\pi}{\cos\frac{\pi p}{2}}\,, F'(p) = \frac{\pi^2\,\sin\frac{\pi p}{2}}{2\,cos^2\,\frac{\pi p}{2}},$$

$$F''(p) = \frac{\pi^3}{4}\frac{cos^2\,\frac{\pi p}{2} + 2sin^2\,\frac{\pi p}{2}}{cos^3\,\frac{\pi p}{2}}\,, F''(0) = \frac{\pi^3}{4}$$

№4. $\int_{-\infty}^{\infty}\frac{x^2}{sh\,x}\,dx = F'(0) = \frac{\pi^2}{2}$

$$F(p) = \int_{-\infty}^{\infty}\frac{sh\,px}{sh\,x}\,dx\,, -1 < p < 1\,, F'(p) = \int_{-\infty}^{\infty}\frac{x\,ch\,px}{sh\,x}\,dx,$$

$$F(p) = \pi\,tg\,\frac{\pi p}{2}\,, F'(p) = \frac{\pi^2}{2\,cos^2\,\frac{\pi p}{2}}\,,\quad F'(0) = \frac{\pi^2}{2}$$

№5. $\int_{0}^{\infty}\frac{\ln x}{(x+1)(x^2+1)}\,dx$=- $F'(0) = -\frac{\pi^2}{16}$

$$F(p) = \int_{0}^{\infty}\frac{x^{1-p}}{\left(x^2\;+1\right)(x+1)}\,dx = \frac{\pi\left(\sin\frac{\pi p}{2} + \cos\frac{\pi p}{2} - 1\right)}{2\sin\pi p} =$$

$$= \frac{-\pi}{2\sin\pi p} + \frac{\pi}{4\cos\frac{\pi p}{2}} + \frac{\pi}{4\sin\frac{\pi p}{2}}$$

$$F'(p) = -\int_0^\infty \frac{x^{1-p}\ln x}{(x^2+1)(x+1)}\,dx = -\frac{\pi^2\cos\frac{\pi p}{2}}{8sin^2\frac{\pi p}{2}} + \frac{\pi^2\sin\frac{\pi p}{2}}{8cos^2\frac{\pi p}{2}} + \frac{\pi^2\cos\pi p}{2sin^2\frac{\pi p}{2}}$$

$$F'(0) = \lim_{p\to 0}\frac{\pi^2}{8sin^2\frac{\pi p}{2}}\left(\frac{\cos\frac{\pi p}{2}}{cos^2\frac{\pi p}{2}} - \cos\frac{\pi p}{2}\right) =$$

$$= \lim_{p\to 0}\frac{\pi^2\left(2cos^2\frac{\pi p}{2} - 1 - \cos\frac{3\pi p}{2}\right)}{8sin^2\frac{\pi p}{2}cos^2\frac{\pi p}{2}}$$

$$= \lim_{p\to 0}\frac{\pi^2\left(1-\cos\frac{\pi p}{2}\right)\left(cos^2\frac{\pi p}{2} - 1 - \cos\frac{\pi p}{2}\right)}{2sin^2\pi p} = -\frac{\pi^2}{16}$$

Семинар 4. Интегралы Эйлера

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt\,, x,y>0$$

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt\,, x>0$$

№1. $\int_0^\infty \frac{ln^2 x}{x^2-x+1}\,dx=F''(0)=\frac{20\pi^3}{81\sqrt{3}}$

$$F(p)=\int_0^\infty \frac{x^p}{x^2-x+1}\,dx\,, -1<p<1\,, F'(p)=\int_0^\infty \frac{x^p\,\ln x}{x^2-x+1}\,dx\,,$$

$$F''(p)=\int_0^\infty \frac{ln^2 x\, x^p}{x^2-x+1}\,dx\,,\qquad F(p)=\int_0^\infty \frac{x^p}{x^3+1}\,dx+\int_0^\infty \frac{x^{p+1}}{x^3+1}\,dx=$$

$$=\frac{\pi}{3\sin\frac{\pi(p+1)}{3}}+\frac{\pi}{3\sin\frac{\pi(p+2)}{3}}=\frac{\pi}{3\sin\frac{\pi(p+1)}{3}}-\frac{\pi}{3\sin\frac{\pi(p-1)}{3}}$$

$$F'(p)=\frac{\pi^2}{9}\left(-\frac{\cos\frac{\pi(p+1)}{3}}{sin^2\frac{\pi(p+1)}{3}}+\frac{\cos\frac{\pi(p-1)}{3}}{sin^2\frac{\pi(p-1)}{3}}\right)$$

$$F''(p)=\frac{\pi^3}{27}\left(\frac{sin^2\frac{\pi(p+1)}{3}+2cos^2\frac{\pi(p+1)}{3}}{sin^3\frac{\pi(p+1)}{3}}-\right.$$
$$\left.-\frac{sin^2\frac{\pi(p-1)}{3}+2cos^2\frac{\pi(p-1)}{3}}{sin^3\frac{\pi(p-1)}{3}}\right)$$

$$F''(0) = \frac{\frac{2\pi^3}{27}\left(\frac{3}{4}+\frac{1}{2}\right)}{\frac{3\sqrt{3}}{8}} = \frac{20\pi^3}{81\sqrt{3}}$$

№2. $\int_0^{\pi/2} sin^2\, x \ln tg\, x\ \ dx = F'(0) = \frac{\pi}{4}$

При любых m, n интеграл $\int_0^{\pi/2} sin^m x\ \ cos^n\, x\, dx$ выражается через бета-функцию, которая не всегда сводится к элементарным функциям. Зато при всех n интеграл $\int_0^{\pi/2} tg^n\, x\ \ \ dx$ представляет собой элементарную функцию.

$$F(p) = \int_0^{\pi/2} sin^2\, x\, tg^p\, x\ \ dx\,, \qquad F'(p) = \int_0^{\pi/2} sin^2\, x\, tg^p\, x\ \ln tg\, x\ \ dx$$

$$F(p) = \int_0^{\pi/2} sin^{2+p} x\, cos^{-p} x\ \ dx = \frac{1}{2}\, B\left(\frac{3+p}{2}, \frac{1-p}{2}\right), -3 < p < 1$$

$$F(p) = \frac{1}{2}\, B\left(\frac{3+p}{2}, \frac{1-p}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3+p}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{\Gamma(2)}$$

$$= \frac{1}{2}\,\frac{\frac{1+p}{2}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{1} = \frac{\pi(p+1)}{4\sin\frac{\pi(1-p)}{2}} = \frac{\pi(p+1)}{4\cos\frac{\pi p}{2}}$$

$$F'(p) = \frac{\pi}{4}\frac{\cos\frac{\pi p}{2} + \frac{\pi(p+1)}{2}\sin\frac{\pi p}{2}}{cos^2\,\frac{\pi p}{2}}\ , \qquad F'(0) = \frac{\pi}{4}$$

№3. $\int_0^\infty \frac{arctg\, x \ln x}{x\sqrt{x}}\, dx = F'\left(\frac{3}{2}\right) = 2\sqrt{2}\pi \quad + \frac{\pi^2\sqrt{2}}{2}$

$$F(p) = \int\limits_0^\infty \frac{arctg\, x}{x^p}\, dx\,, 1 < p < 2\,, F'(p) = \int\limits_0^\infty \frac{arctg\, x\ \ln x}{x^p}\, dx\,,$$

$$F(p) = \frac{\pi}{2(p-1)\sin\frac{\pi p}{2}}\,, F'(p) = -\frac{\pi}{2}\ \left(\frac{\sin\frac{\pi p}{2} + \frac{\pi}{2}\cos\frac{\pi p}{2}(p-1)}{(p-1)^2\ sin^2\ \frac{\pi p}{2}}\right)$$

$$F'\left(\frac{3}{2}\right) = \frac{\pi}{4}\ \frac{\frac{\sqrt{2}}{4} - \frac{\pi\ \sqrt{2}}{8}}{\frac{1}{4}\ \frac{1}{2}} = 2\sqrt{2}\pi \quad + \frac{\pi^2\sqrt{2}}{2}$$

Формулы для вычисления производной напрямую.

I. $\Gamma'\ (x) = \int_0^\infty t^{x-1}e^{-t}\ln t\, dt$

$$\Gamma'\ (1) = \int\limits_0^\infty e^{-t}\ln t\, dt = \ -\gamma$$

$$e^{-t} = \lim_{n\to\infty}(1 - \frac{t}{n})^n\ , t\ \in [0; n]$$

$$\int\limits_0^\infty e^{-t}\ln t\, dt = \lim_{n\to\infty}\int\limits_0^n (1 - \frac{t}{n})^n\ \ln t\, dt = -\gamma$$

$$\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - ln\, n\ )$$

II. $\Gamma''\,(x) = \int_0^\infty t^{x-1}e^{-t}ln^2\ t\ dt$

$$\Gamma'\ (1) = \int_0^{\infty} e^{-t} ln^2\ tdt = \gamma^{\ 2}\ + \frac{\pi^2}{6}$$

$$e^{-t} = \lim_{n\to\infty} (1 - \frac{t}{n})^n\ , t\ \in [0; n]$$

$$\int_0^{\infty} e^{-t} ln^2\ tdt = \lim_{n\to\infty} \int_0^{n} (1 - \frac{t}{n})^n\ ln^2\ tdt = \gamma^{\ 2}\ + \frac{\pi^2}{6}$$

$$\gamma = \lim_{n\to\infty} (\sum_{k=1}^{n} \frac{1}{k} - ln\ n\ )$$

№4. $\int_0^{\infty} \frac{\sin x \ln x}{x}\ dx = -F'(1) = 2\sqrt{2}\pi \quad + \frac{\pi^2\sqrt{2}}{2}$

$$F(p) = \int_0^{\infty} \frac{\sin x}{x^p}\ dx\ , 0 < p < 2\ , F'(p) = -\int_0^{\infty} \frac{\sin x\ \ \ln x}{x^p}\ dx\ ,$$

$$F(p) = \frac{\pi}{2\Gamma(p) \sin\frac{\pi p}{2}}\ , F'(p) = -\frac{\pi}{2}\ (\frac{\Gamma'\ (p) \sin\frac{\pi p}{2} + \frac{\pi}{2}\cos\frac{\pi p}{2}\Gamma\ \ (p)}{\Gamma^2\ (p) sin^2\ \frac{\pi p}{2}})$$

$$-F'(1) = \frac{-\gamma\pi}{2}\ , F''(p) = \int_0^{\infty} \frac{\sin x\ \ ln^2\ x}{x^p}\ dx\ , F''(1) = \int_0^{\infty} \frac{\sin x\ \ ln^2\ x}{x}\ dx$$

$$F''(p) = -\frac{\pi}{2}\left(\frac{\Gamma'\ (p) \sin\frac{\pi p}{2} + \frac{\pi}{2}\cos\frac{\pi p}{2}\Gamma\ \ (p)}{\Gamma^2\ (p) sin^2\ \frac{\pi p}{2}}\right)'$$

$$= -\frac{\pi}{2}\,\frac{\left(\Gamma''(p)\Gamma\ \ (p)\sin\frac{\pi p}{2} - 2(\Gamma'\ (p))^2\ (p)\sin\frac{\pi p}{2} - \frac{\pi}{2}\cos\frac{\pi p}{2}\Gamma\ \ (p)\Gamma'\ (p)\right)}{\Gamma^3\ (p) sin^2\ \frac{\pi p}{2}}$$

$+$

$$+\frac{\pi}{2}\,\frac{\frac{-\pi}{2} sin^2\ \frac{\pi p}{2}\Gamma\ \ (p) - \cos\frac{\pi p}{2}\sin\frac{\pi p}{2}\Gamma'\ (p))}{\Gamma^2\ (p) sin^3\ \frac{\pi p}{2}},$$

$$F''(1) = -\frac{\pi}{2}\ (\gamma^{\,2} + \frac{\pi^2}{6} - 2\gamma^{\,2} - \frac{\pi^2}{4})$$

Семинар 5. Производные гамма-функции (часть I)

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, x > 0$$

$$\Gamma'(x) = \int_0^{\infty} t^{x-1} e^{-t} \ln t \, dt$$

$$\Gamma'(1) = \int_0^{\infty} e^{-t} \ln t \, dt = -\gamma$$

$$\Gamma(x+1) = x\,\Gamma(x), \qquad \Gamma'(x+1) = \Gamma(x) + x\Gamma'(x), \qquad \Gamma'(2) = 1 - \gamma$$

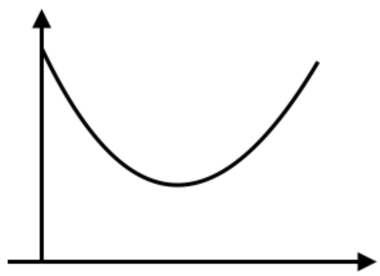

Рис. 5.1. График гамма-функции.

№1. $\int_0^{\pi/2} \ln\sin x \, dx = \begin{cases} \int_{\pi/2}^{\pi} \ln\sin t \, dt, x = \pi - t \\ \int_0^{\pi/2} \ln\cos t \, dt, x = \frac{\pi}{2} - t \end{cases} = \frac{-\pi}{2} \ln 2 = F'(0)$

$$\int_0^{\pi/2} \ln\sin t \, dt = \frac{1}{2}\int_0^{\pi} \ln\sin t \, dt, \qquad \int_0^{\pi/2} \ln\sin t \, dt = \int_0^{\pi/2} \ln\cos t \, dt$$

$$\int_0^{\pi/2} \ln\cos x\ dx = \frac{1}{2}\int_0^{\pi/2} \ln\sin x\cos x\ dx = \frac{1}{2}\int_0^{\pi/2} \ln\frac{1}{2}\sin 2x\ dx = |\ 2x = t|$$

$$= \frac{1}{4}\int_0^{\pi/2} \ln\frac{1}{2}\sin t\ dt = \frac{-\pi}{4}\ln 2 + \frac{1}{4}\int_0^{\pi/2} \ln\sin t\ dt = \frac{-\pi}{2}\ln 2$$

$$F\quad(p) = \int_0^{\pi/2} sin^p\, x\ \ dx\,, p > 1, F'(p) = \int_0^{\pi/2} sin^p\, x \ln\sin x\ dx$$

$$F\quad(p) = \frac{1}{2}\ B\left(\frac{p+1}{2}\ , \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(\frac{1}{2})}{\Gamma(\frac{p}{2}+1)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(\frac{p}{2}+1)}$$

$$F'(p) = \frac{\sqrt{\pi}}{2}\frac{\Gamma'\ (\frac{p}{2}+1)(\frac{p+1}{2}) - \Gamma'\ (\frac{p}{2}+1)\Gamma(\frac{p+1}{2})}{\Gamma^3\ (\frac{p}{2}+1)}$$

$$F'(0) = \frac{\sqrt{\pi}}{4}\left(\Gamma'\ \left(\frac{1}{2}\right) + \gamma\sqrt{\pi}\right) = \frac{-\pi}{2}\ln 2\,, \Gamma'\ \left(\frac{1}{2}\right) = -\gamma\sqrt{\pi} - 2\ln 2\sqrt{\pi}$$

$$\Gamma'\ \left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\ln 2),$$

$$\Gamma'\ \left(\frac{3}{2}\right) = \sqrt{\pi} - \frac{\sqrt{\pi}}{2}(\gamma + 2\ln 2) = \sqrt{\pi}\left(1 - \frac{\gamma}{2} - \ln 2\right) \approx 3 * 10^{-2} > 0$$

№2. $\int_0^1 \frac{t^{-p} - t^{p-1}}{1-t}\ dt\,, 0 < p < 1$

$F(\varepsilon) = \int_0^1 \frac{t^{-p} - t^{p-1}}{(1-t)^{1-\varepsilon}}\, dt = \lim_{\varepsilon \to +0} F(\varepsilon) = \pi\, ctg\, \pi p$

№3. $\int_0^1 \frac{x^p}{x^2+x+1}\, dx = \int_0^1 \frac{x^p - x^{p+1}}{1-x^3}\, dx = |x^3 = t| = \frac{1}{3}\int_0^1 \frac{t^{\frac{p-2}{3}} - t^{\frac{p-1}{3}}}{1-t}\, dt$

$$\int_0^\infty \frac{x^p}{x^2+x+1}\, dx = \int_0^1 \frac{x^p}{x^2+x+1}\, dx + \int_1^\infty \frac{x^p}{x^2+x+1}\, dx$$

$$= \int_0^1 \frac{x^p}{x^2+x+1}\, dx - \int_1^0 \frac{t^{-p}}{t^{-2}+t^{-1}+1}\frac{dt}{t^2}$$

$$= \int_0^1 \frac{x^p}{x^2+x+1}\, dx + \int_0^1 \frac{x^{-p}}{x^2+x+1}\, dx$$

$$\int_0^\infty \frac{x^p}{x^2+x+1}\, dx = \frac{1}{3}\int_0^1 \frac{t^{\frac{p-2}{3}} - t^{\frac{p-1}{3}}}{1-t}\, dt + \frac{1}{3}\int_0^1 \frac{t^{-\frac{p-2}{3}} - t^{-\frac{p-1}{3}}}{1-t}\, dt =$$

$$= \frac{1}{3}\int_0^1 \frac{t^{\frac{p-2}{3}} - t^{-\frac{p-1}{3}}}{1-t}\, dt + \frac{1}{3}\int_0^1 \frac{t^{-\frac{p-2}{3}} - t^{\frac{p-1}{3}}}{1-t}\, dt$$

$$= \frac{\pi}{3}\, ctg\, \frac{\pi(p-2)}{3} + \frac{\pi}{3}\, ctg\, \frac{\pi(p+2)}{3}$$

№4. $\int_{-\infty}^{\infty} \frac{sh\, \alpha x}{sh\, \beta x}\, dx, 0 < \alpha < \beta$

$$\int_{-\infty}^{\infty} \frac{sh\, \alpha x}{sh\, \beta x}\, dx = \int_{-\infty}^{\infty} \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\beta x} - e^{-\beta x}}\, dx = \left| e^{\beta x} = t \right| = \pi\, tg\, \frac{\pi\alpha}{2\beta}$$

№5. $\int_0^\infty \frac{e^{-x^p}-e^{-x^q}}{x}\,dx = \lim_{\varepsilon\to+0} F(\varepsilon)\,, p,q>0$

$$F(\varepsilon) = \int_0^\infty \frac{e^{-x^p}-e^{-x^q}}{x^{1-\varepsilon}}\,dx = \int_0^\infty \frac{e^{-x^p}}{x^{1-\varepsilon}}\,dx \int_0^\infty \frac{e^{-x^q}}{x^{1-\varepsilon}}\,dx$$

$$= \frac{1}{p}\int_0^\infty t^{\frac{1-\varepsilon}{p}} e^{-t} t^{\frac{1}{p}-1}\,dt - \frac{1}{q}\int_0^\infty t^{\frac{\varepsilon-1}{q}} e^{-t} t^{\frac{1}{q}+1}\,dt$$

$$= \frac{1}{p}\Gamma\left(\frac{\varepsilon}{p}\right) - \frac{1}{q}\Gamma\left(\frac{\varepsilon}{q}\right) = \frac{1}{\varepsilon} - \frac{\gamma}{p} + O(\varepsilon) - \frac{1}{\varepsilon} - \frac{\gamma}{q} + O(\varepsilon)$$

$$\Gamma(x) = \frac{\Gamma(1+x)}{p} = \frac{1-\gamma x + O(x^2)}{x}, \qquad F(x) = \frac{1}{x} - \gamma + O(x)$$

№6. $\int_0^\infty \frac{\cos x^p - \cos x^q}{x}\,dx = \lim_{\varepsilon\to+0} F(\varepsilon) = \gamma(\frac{1}{q}-\frac{1}{p}), p,q>0$

$$F(\varepsilon) = \int_0^\infty \frac{\cos x^p - \cos x^q}{x^{1-\varepsilon}}\,dx = \int_0^\infty \frac{\cos x^p}{x^{1-\varepsilon}}\,dx - \int_0^\infty \frac{\cos x^q}{x^{1-\varepsilon}}\,dx$$

$$= \frac{1}{p}\int_0^\infty t^{\frac{\varepsilon}{p}-1}\cos t\;dt - \frac{1}{q}\int_0^\infty t^{\frac{\varepsilon}{q}-1}\cos t\;dt =$$

$$= \frac{1}{2p}\frac{\pi}{\Gamma\left(1-\frac{\varepsilon}{p}\right)\cos\frac{\pi}{2}\left(1-\frac{\varepsilon}{p}\right)} - \frac{1}{2q}\frac{\pi}{\Gamma\left(1-\frac{\varepsilon}{q}\right)\cos\frac{\pi}{2}\left(1-\frac{\varepsilon}{q}\right)}$$

$$= \left|\Gamma(1-x) = \frac{\pi}{\Gamma(x)\sin\pi x}\right| =$$

$$= \frac{1}{2p}\frac{\Gamma\left(\frac{\varepsilon}{p}\right)\sin\frac{\varepsilon}{p}}{\sin\frac{\pi\varepsilon}{2p}} - \frac{1}{2q}\frac{\Gamma\left(\frac{\varepsilon}{q}\right)\sin\frac{\varepsilon}{q}}{\sin\frac{\pi\varepsilon}{2q}} = \frac{1}{p}\Gamma\left(\frac{\varepsilon}{p}\right)\cos\frac{\pi\varepsilon}{2p} - \frac{1}{q}\Gamma\left(\frac{\varepsilon}{q}\right)\cos\frac{\pi\varepsilon}{2q}$$

Семинар 6. Производные гамма-функции (часть II)

№1. $\int_0^\infty \frac{e^{-x^p} - \cos x^q}{x}\,dx = \lim\limits_{\varepsilon\to+0} F(\varepsilon) = \gamma\left(\frac{1}{q} - \frac{1}{p}\right)$

$$F(\varepsilon) = \int\limits_0^\infty \frac{e^{x^p} - \cos x^q}{x^{1-\varepsilon}}\,dx = \int\limits_0^\infty \frac{e^{-x^p}}{x^{1-\varepsilon}}\,dx - \int\limits_0^\infty \frac{\cos x^q}{x^{1-\varepsilon}}\,dx$$

$$= \frac{1}{p}\int\limits_0^\infty t^{\frac{\varepsilon}{p}-1} e^{-t}\,dt - \frac{1}{q}\int\limits_0^\infty \frac{\cos x^q}{t^{1-\frac{\varepsilon}{q}}}\,dt =$$

$$= \frac{1}{p}\Gamma\left(\frac{\varepsilon}{p}\right) - \frac{1}{2q}\frac{\pi}{\Gamma\left(1-\frac{\varepsilon}{q}\right) cos\frac{\pi}{2}\left(1-\frac{\varepsilon}{q}\right)}$$

$$= \frac{1}{p}\Gamma\left(\frac{\varepsilon}{p}\right) - \frac{\Gamma\left(\frac{\varepsilon}{p}\right)\sin\frac{\pi\varepsilon}{p}}{2q\sin\frac{\pi\varepsilon}{2p}} = \frac{1}{p}\Gamma\left(\frac{\varepsilon}{p}\right) - \frac{1}{q}\Gamma\left(\frac{\varepsilon}{q}\right)\cos\frac{\pi\varepsilon}{2p} =$$

$$= \frac{1}{p}\left(\frac{p}{\varepsilon} - \gamma + O(\varepsilon)\right) - \frac{1}{q}\left(\frac{q}{\varepsilon} - \gamma + O(\varepsilon)\right)\left(1 + O\left(\varepsilon^2\right)\right)$$

$$= \gamma\left(\frac{1}{q} - \frac{1}{p}\right) + O(\varepsilon)$$

№2. $\int_0^1 \frac{1-e^{-x}}{x}\,dx - \int_1^\infty \frac{e^{-x}}{x}\,dx \neq \int_0^1 \frac{1}{x}\,dx - \int_0^\infty \frac{e^{-x}}{x}\,dx$

$$\int\limits_0^1 \frac{1-e^{-x}}{x}\,dx - \int\limits_1^\infty \frac{e^{-x}}{x}\,dx = \lim_{\varepsilon\to+0} F(\varepsilon) = \gamma$$

$$F(\varepsilon) = \int_0^1 \frac{1-e^{-x}}{x^{1-\varepsilon}}\,dx - \int_0^\infty \frac{e^{-x}}{x^{1-\varepsilon}}\,dx = \int_0^1 \frac{1}{x^{1-\varepsilon}}\,dx - \int_0^\infty \frac{e^{-x}}{x^{1-\varepsilon}}\,dx =$$

$$= \frac{1}{\varepsilon} - \frac{1}{\varepsilon} + \gamma + O(\varepsilon)$$

№3. $\int_0^1 \frac{1-\cos x}{x}\,dx - \int_1^\infty \frac{\cos x}{x}\,dx = \lim\limits_{\varepsilon\to+0} F(\varepsilon) =$

$$F(\varepsilon) = \int_0^1 \frac{1-\cos x}{x^{1-\varepsilon}}\,dx - \int_0^\infty \frac{\cos x}{x^{1-\varepsilon}}\,dx = \int_0^1 \frac{1}{x^{1-\varepsilon}}\,dx - \int_0^\infty \frac{\cos x}{x^{1-\varepsilon}}\,dx =$$

$$= \frac{1}{\varepsilon} - \frac{\pi}{2\Gamma(1-\varepsilon)\cos\frac{\pi(1-\varepsilon)}{2}} = \frac{1}{\varepsilon} - \Gamma(\varepsilon)\cos\frac{\pi\varepsilon}{2} =$$

$$= \frac{1}{\varepsilon} - \left(\frac{1}{\varepsilon} - \gamma + O(\varepsilon)\right)\left(1 + O\left(\varepsilon^2\right)\right) = \gamma + O(\varepsilon)$$

$$\Gamma'(1) = -\gamma, \qquad \Gamma''(1) = \gamma^2 + \frac{\pi^2}{6}, \qquad \Gamma'\left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\ln 2)$$

№4. $\int_0^{\frac{\pi}{2}} \frac{\ln\sin x}{\cos x}\,dx = \left|\begin{matrix} \cos x = \cos\left(\frac{\pi}{2} - t\right) = \sin t,\ t = \frac{\pi}{2} - x,\ t \to +0 \\ \ln\sin x = \ln\sin\left(\frac{\pi}{2} - t\right) = \ln\cos t,\ -\frac{t^2}{2} = -\frac{(\frac{\pi}{2}-x)^2}{2} \end{matrix}\right| =$

$-\frac{\pi^2}{8}$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x\ \ln\sin x}{cos^2 x}\,dx = |\sin x = t| = \int_0^1 \frac{\ln t}{1-t^2}\,dt = \int_0^1 \Big(\sum_{n=0}^{\infty} t^{2n}\ln t\Big)\,dt$$

$$= \sum_{n=0}^{\infty}\int_0^1 t^{2n}\ln t\,dt = \frac{1}{2}\sum_{n=0}^{\infty}\frac{d}{dn}\Big(\int_0^1 t^{2n}\,dt\Big) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{d}{dn}\frac{1}{2n+1}$$

$$= -\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = -\frac{\pi^2}{8}$$

$$\int_0^1 \frac{\ln t}{1-t^2}\,dt = \lim_{\varepsilon\to+0}\int_0^{1-\varepsilon} \frac{\ln t}{1-t^2}\,dt = \lim_{\varepsilon\to+0}\sum_{n=0}^{\infty}\int_0^{1-\varepsilon} t^{2n}\ln t\;dt$$
$$= \sum_{n=0}^{\infty}\lim_{\varepsilon\to+0}\int_0^{1-\varepsilon} t^{2n}\ln t\;dt$$

$$\int_0^{\frac{\pi}{2}} \frac{\ln\sin x}{\cos x}\,dx =$$

$F(p) = \int_0^{\frac{\pi}{2}} \frac{sin^p\, x}{\cos x}\,dx\,,\ F'(p) = \int_0^{\frac{\pi}{2}} \frac{sin^p\, x\ln\sin x}{\cos x}\,dx$ – не годится

$$G\quad(\varepsilon) = \int_0^{\frac{\pi}{2}} \frac{\ln\sin x}{cos^{1-\varepsilon}x}\,dx\,, \qquad H_\varepsilon\ (p) = \int_0^{\frac{\pi}{2}} \frac{sin^p\ x}{cos^{1-\varepsilon}x}\,dx\,,$$

$$\frac{d}{dp}H_\varepsilon\ (p) = \int_0^{\frac{\pi}{2}} \frac{sin^p\ x ln\sin x}{cos^{1-\varepsilon}x}\,dx\,, \frac{d}{dp}\Big|_{p\,=\,0} H_\varepsilon\ (p) = G\quad(\varepsilon)$$

$$H_\varepsilon\ (p) = \frac{1}{2}\ B\left(\frac{p+1}{2},\frac{\varepsilon}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(\frac{\varepsilon}{2})}{2\ \Gamma(\frac{p+1+\varepsilon}{2})}$$

$$\frac{d}{dp}H_\varepsilon\ (p) = \frac{\Gamma(\frac{\varepsilon}{2})}{4}\ \frac{\Gamma\left(\frac{p+1+\varepsilon}{2}\right)\Gamma'\ \ (\frac{p+1}{2}) - \Gamma'\ \ (\frac{p+1}{2})\Gamma(\frac{p+1}{2})}{\Gamma^2\ (\frac{p+1+\varepsilon}{2})}$$

$$\lim_{\varepsilon\to+0} G(\varepsilon) = \frac{1}{4\pi}\lim_{\varepsilon\to+0}\frac{\Gamma\left(\frac{1+\varepsilon}{2}\right)\Gamma'\left(\frac{1}{2}\right)-\Gamma'\left(\frac{1+\varepsilon}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{\varepsilon}{2}}$$

$$= \frac{1}{2\pi}\lim_{\varepsilon\to+0}\frac{\Gamma'\left(\frac{1}{2}\right)\left(\Gamma\left(\frac{1}{2}\right)+\Gamma'\left(\frac{1}{2}\right)\frac{\varepsilon}{2}+o(\varepsilon^2)\right)-\Gamma\left(\frac{1}{2}\right)\left(\Gamma'\left(\frac{1}{2}\right)+\Gamma''\left(\frac{1}{2}\right)\frac{\varepsilon}{2}+o(\varepsilon^2)\right)}{\varepsilon}$$

$$= \frac{1}{2\pi}\left(\left(\Gamma'\left(\frac{1}{2}\right)\right)^2 - \Gamma\left(\frac{1}{2}\right)\Gamma''\left(\frac{1}{2}\right)\right)$$

$$-\frac{\pi^2}{8} = \frac{1}{4\pi}\left(\left(\Gamma'\left(\frac{1}{2}\right)\right)^2 - \Gamma\left(\frac{1}{2}\right)\Gamma''\left(\frac{1}{2}\right)\right)$$

$$\Gamma''\left(\frac{1}{2}\right) = \frac{\pi^{5/2}}{2} + \sqrt{\pi}(\gamma + 2\ln 2)^2$$

Семинар 7. Производные гамма-функции (часть III)

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\,, x > 0, \Gamma'\ (x) = \int_0^\infty t^{x-1}e^{-t}\ln t\, dt\,, \Gamma'\ (x) = \int_0^\infty t^{x-1}e^{-t}ln^2\, t\, dt$$

$$\Gamma'\ (1) = -\gamma, \qquad \Gamma'\ \left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\ln 2),$$

$$\Gamma''\,(1) = \gamma^{\ 2} + \frac{\pi^2}{6}\,, \qquad \Gamma''\left(\frac{1}{2}\right) = \frac{\pi^{5/2}}{2} + \sqrt{\pi}(\gamma + 2\ln 2)^2$$

№1. $\int_0^\infty \frac{\sin x \ln x}{\sqrt{x}}\ dx$

$$F\ \ (p) = \int_0^\infty \frac{\sin x}{x^p}\ \ dx\,, F'(p) = \int_0^\infty \frac{\sin x}{x^p}\ln x\ \ dx\,, 0 < p < 2$$

$$F\ \ (p) = \frac{-\pi}{2\Gamma(p)\sin\frac{\pi p}{2}}\,, F'(p) = \frac{-\pi}{2}\ \frac{\Gamma'\ (p)\sin\frac{\pi p}{2} + \frac{\pi}{2}\ \Gamma\ \ (p)\cos\frac{\pi p}{2}}{\Gamma^2\ (p)(\sin\frac{\pi p}{2})^2}$$

$$F'\ \left(\frac{1}{2}\right) = \frac{\pi}{2}\ \frac{\left(-\sqrt{\pi}(\gamma + 2\ln 2)\ \ + \frac{\pi^{\frac{3}{2}}}{2}\right)\frac{\sqrt{2}}{2}}{\frac{\pi}{2}} = \frac{\sqrt{2\pi}}{2}\left(\frac{\pi}{2} - \gamma - 2\ln 2\right)$$

№1. $\int_{-\infty}^\infty \frac{1}{ch\, x\,(x^2 + \pi^2\,)}\ dx = F(0) = \frac{4}{\pi} - 1$

$$F(y) = \int\limits_{-\infty}^{\infty} \frac{ch\, yx}{ch\, x\left(x^2 + \pi^2\right)}\, dx,$$

$$ch\, yx \le ch\, x\,, |y| \le 1, \frac{ch\, yx}{ch\, x\left(x^2 + \pi^2\right)} \le \frac{1}{\left(x^2 + \pi^2\right)}, F(y) \in C[-1;1]$$

$$F'(y) = \int\limits_{-\infty}^{\infty} \frac{x\, sh\, yx}{ch\, x\left(x^2 + \pi^2\right)}\, dx\,, F(y) \in C[-1;1]$$

$$F''(y) = \int\limits_{-\infty}^{\infty} \frac{x^2\; sh\, yx}{ch\, x\left(x^2 + \pi^2\right)}\, dx = \int\limits_{-\infty}^{\infty} \frac{ch\, yx}{ch\, x}\, dx - \int\limits_{-\infty}^{\infty} \frac{\pi^2\; ch\, yx}{ch\, x\left(x^2 + \pi^2\right)}\, dx$$

$$F''(y) + \pi^2 F(y) = \int\limits_{-\infty}^{\infty} \frac{ch\, yx}{ch\, x}\, dx = \frac{\pi}{cos\frac{\pi y}{2}}\;, F(-y) = F(y), F(1) = 1$$

$$F(y) = C_1\;(y)\cos\pi y + C_2\;(y)\sin\pi y \Longrightarrow \begin{cases} C_1'(y)\cos\pi y + C_2'(y)\sin\pi y = 0 \\ -\pi\, C_1'(y)\sin\pi y + \pi C_2'(y)\cos\pi y = \frac{\pi}{cos\frac{\pi y}{2}} \end{cases}$$

$$C_1'(y) = -\frac{\sin\pi y}{cos\frac{\pi y}{2}} = -2\sin\frac{\pi y}{2} \;\Longrightarrow\; C_1\;(y) = \frac{4}{\pi}cos\frac{\pi y}{2} + A$$

$$C_2'(y) = \frac{\cos\pi y}{cos\frac{\pi y}{2}} = 2\cos\frac{\pi y}{2} - \frac{1}{\cos\frac{\pi y}{2}} \;\Longrightarrow\; C_1\;(y) = \frac{4}{\pi}sin\frac{\pi y}{2} - \frac{1}{\pi}\ln\frac{1+\sin\frac{\pi y}{2}}{1-\sin\frac{\pi y}{2}} + B$$

$$F(y) = \left(\frac{4}{\pi}cos\frac{\pi y}{2} + A\right)\cos\pi y + \left(\frac{4}{\pi}sin\frac{\pi y}{2} - \frac{1}{\pi}\ln\frac{1+\sin\frac{\pi y}{2}}{1-\sin\frac{\pi y}{2}} + B\right)\sin\pi y\,, B = 0$$

$F(-1) = -A \;\Longrightarrow A = 1$

$$F(y) = \left(\frac{4}{\pi}cos\frac{\pi y}{2} + 1\right)\cos\pi y + \left(\frac{4}{\pi}sin\frac{\pi y}{2} - \frac{1}{\pi}\ln\frac{1+\sin\frac{\pi y}{2}}{1-\sin\frac{\pi y}{2}}\right)$$

Второй способ:

$$\frac{1}{\left(x^2+\pi^2\right)}=\frac{1}{\pi}\int_0^\infty e^{-\pi y}\cos yx\ dy$$

$$\int_{-\infty}^{\infty}\frac{1}{\left(x^2+\pi^2\right)\ ch\,x}\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}dx\int_0^\infty e^{-\pi y}\cos yx\ dx=$$

$$=\left|\left|e^{-\pi y}\frac{1}{\left(x^2+\pi^2\right)\ ch\,x}\right|\le\frac{e^{-\pi y}}{ch\,x}\right|=\frac{1}{\pi}\int_0^\infty e^{-\pi y}dy\int_{-\infty}^{\infty}e^{-\pi y}\frac{\cos yx}{ch\,x}\,dx$$

$$=\frac{1}{\pi}\int_0^\infty e^{-\pi y}\frac{\pi}{\operatorname{ch}\frac{\pi y}{2}}dy=\left|\begin{matrix}e^{-\frac{\pi y}{2}}=t\\ y=\frac{2}{\pi}\ln t\end{matrix}\right|=\frac{4}{\pi}\int_0^\infty\frac{t^2}{\left(t+\frac{1}{t}\right)t}dt$$

$$=\frac{4}{\pi}\int_0^\infty\frac{t^2}{t^2+1}dt=\frac{4}{\pi}-1$$

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