МЕХАНИКО-МАТЕМАТИЧЕСКИЙ ФАКУЛЬТЕТ МГУ ИМЕНИ М.В. ЛОМОНОСОВА

teach-in
ЛЕКЦИИ УЧЕНЫХ МГУ

ВВЕДЕНИЕ В КВАНТОВЫЕ ИНТЕГРИРУЕМЫЕ СИСТЕМЫ

ЗОТОВ
АНДРЕЙ ВЛАДИМИРОВИЧ

МЕХМАТ МГУ

БЛАГОДАРИМ ЗА ПОДГОТОВКУ КОНСПЕКТА
СТУДЕНТА МЕХАНИКО-МАТЕМАТИЧЕСКОГО ФАКУЛЬТЕТА МГУ
РАЕВА АЛЕКСЕЯ АЛЕКСЕЕВИЧА

Zotov Andrei Vladimirovich

Lecture notes

Quantum integrable systems

Lecture notes were prepared by the student of the mathematics and mechanics faculty
Raev Aleksey Alekseevich

Contents

Lecture 1

Classical integrable systems

In this course we will consider 2 kinds of integrable systems:

1) Many-body systems,

2) spin systems.

We will focus on the Bethe ansatz method as soon as we approach the quantum integrability.

The classical mechanics involves 2 main objects:

1) Phase space, namely, some (Poisson) manifold M,

2) Hamilton function $H \in C^\infty(M)$.

The dynamics of the system is described by the following equation (we denote $\dot{f} = \frac{df}{dt}$)

$$\dot{f} = \{H, f\},$$

where

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}.$$

We also assume that $\{p_i, q_j\} = \delta_{ij}$.
Quantisation of a classical system implies that we introduce (for each function f) $\hat{f} \in \mathrm{End}(V)$ an operator acting on some Hilbert space of states and we change Poisson bracket to a commutator, where $\hbar$ arises. By taking limit $\hbar \to 0$, we arrive to a classical system.

Theorem 1 (Liouville)**.** *The system of N particles is called integrable, if there exist N independent integrals of motion (conservation laws) in involution:*

$$\{I_i, I_k\} = 0.$$

In terms of quantum mechanics, we get

$$\{f, g\} = \lim_{\hbar \to 0} \frac{\hat{f}\hat{g} - \hat{g}\hat{f}}{\hbar}.$$

Lax representation

Consider n-particles system described by $2n$ equations of motion:

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i}, \\ \dot{q}_i = \frac{\partial H}{\partial p_i}. \end{cases}$$

Let us now introduce the Lax pair.

Definition 1. *The equation, written for* $L, M \in \mathrm{Mat}_N$

$$\dot{L} = [L, M] \tag{1}$$

is called the Lax pair for L, M.

Remark 1. *From (1) we derive conserved quantities*

$$H_k = \frac{1}{k}\operatorname{tr}(L^k), \;\; k \in \mathbb{Z}_+.$$

Proof. Let us prove it by the induction on k: if $k = 1$, then $\operatorname{tr} L = \operatorname{tr}(LM - ML) = 0$. For $k = 2$ we have

$$\frac{1}{2}\frac{d}{dt}\operatorname{tr}(L^2) = \frac{1}{2}\operatorname{tr}(\dot{L}L + L\dot{L}) = \operatorname{tr}(\dot{L}L) = \operatorname{tr}([L, M]L) = \operatorname{tr}(LML - ML^2) = 0.$$

Then the induction step is evident. □

Example [Calogero-Moser model].
The model is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{i}^{N} p_i^2 - \frac{\nu^2}{2}\sum_{i \neq j}^{N}\frac{1}{(q_i - q_j)^2}, \tag{2}$$

where ν is a coupling constant. Let us introduce the $N \times N$ matrix given by

$$L_{ij} = \delta_{ij}\dot{q}_i + (1 - \delta_{ij})\frac{\nu}{q_i - q_j}. \tag{3}$$

Then for the same size matrix $M = \delta_{ij}d_i - (1 - \delta_{ij})\frac{\nu}{(q_i - q_j)^2}$ we have

$$\dot{L} = [L, M], \tag{4}$$

where $d_i = \nu \sum\limits_{i \neq k} \frac{1}{(q_i - q_j)^2}$. The main property is that we have the involution

$$\{H_k, H_m\} = 0. \tag{5}$$

Classical r-matrix

Denote $L_1 := L \otimes I_N$ and $L_2 = I_n \otimes L$. Let us recall tensor product of two matrices first. For given $A, B \in \mathrm{Mat}_N$ we have

$$A \otimes B := \begin{pmatrix} A_{11}B & A_{12}B & \dots & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & \dots & A_{2n}B \\ \dots & \dots & \dots & \dots & \dots \\ A_{n1}B & \dots & \dots & \dots & A_{nn}B. \end{pmatrix} \tag{6}$$

The resulting matrix has the size $N^2 \times N^2$. Consider now some properties of this tensor operation:

1) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. This property is easily derived from the definition of the tensor product. Indeed,

$$A_{11}BC_{11}D + A_{12}BC_{21}D + \dots = \sum_{k=1}^{N} A_{1k}C_{k1}BD. \tag{7}$$

2) $(A \otimes B)(u \otimes v) = (Au \otimes Bv)$, where $u, v \in \mathbb{C}^n$.

Let us introduce the standard notation for the basis in Mat_N : $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$, where E_{ij} is the matrix with 1 at i-th row and j-th column. Hence, any matrix can be decomposed via this standard basis

$$A = \sum_{i,j=1}^{N} E_{ij}A_{ij}. \tag{8}$$

In $\mathrm{Mat}_N^{\otimes 2}$ we can introduce the basis $E_{ij} \otimes E_{kl}$. Then we can write

$$r_{12} = \sum_{ij,kl=1}^{N} r_{ijkl}E_{ij} \otimes E_{kl}. \tag{9}$$

Equivalently,

$$r_{21} = \sum_{ij,kl=1}^{N} r_{ijkl}E_{kl} \otimes E_{ij}. \tag{10}$$

We can now define Poisson brackets $\{L_{ij}, L_{kl}\}$, then

$$\{L \otimes L\} = \{L_1, L_2\} = \sum_{ij,kl=1}^{N} \{L_{ij}, L_{kl}\}E_{ij} \otimes E_{kl}. \tag{11}$$

Proposition 1. *Suppose there exists*

$$r_{12} = \sum_{ij,kl=1}^{N} r_{ijkl}E_{ij} \otimes E_{kl} \tag{12}$$

such that

$$\{L_1, L_2\} = [L_1, r_{12}] - [L_2, r_{21}]. \tag{13}$$

Then

$$\{H_k, H_m\} = 0,\ \ H_k = \frac{1}{k}\,\mathrm{tr}(L^k). \tag{14}$$

The equation (12) defines the classical r-matrix and (13) defines classical r-matrix structure, which is a criterion for the Liouville integrability. In the formalism of quantum mechanics we will deal with the quantum analogues of these objects.
For the Calogero-Moser model r-matrix has the form

$$r_{12} = \sum_{i \neq j} E_{ij} \otimes E_{ji}\frac{1}{q_i - q_j} + E_{ij} \otimes E_{jj}\frac{1}{q_i - q_j}. \tag{15}$$

The equation for the r-matrix follows from Jacobi identity

$$\{\{f,g\},h\} + cycl = 0. \tag{16}$$

So, we obtain

$$\{\{L_{ij}, L_{kl}\}, L_{mn}\} + cycl = 0 \tag{17}$$

and $\mathrm{Mat}_N^{\otimes 3} \in E_{ij} \otimes E_{kl} \otimes E_{mn}$. Finally, for

$$L_1 := L \otimes 1 \otimes 1,$$

$$L_2 := 1 \otimes L \otimes 1,$$

$$L_3 := 1 \otimes 1 \otimes L$$

we obtain

$$\{\{L_1, L_2\}L_3\} + \{\{L_2, L_3\}L_1\} + \{\{L_3, L_1\}L_2\} = 0. \tag{18}$$

This Jacobi identity is a sufficient condition for the classical Yang-Baxter equation. When r-matrix is skew-symmetric and nondynamical ($\{r_{12}, L_3\} = 0$), $r_{12} = -r_{21}$ we have

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{19}$$

The Yang-Baxter equation can be inhomogeneous in some cases.
If we have the Lax equation, then

$$\frac{d}{dt_k} = [L, M_k],\ M_k = -\operatorname{tr}_2(r_{12}L_2^{k-1}) \tag{20}$$

and k corresponds to different Hamilton functions. We have to compute the trace:

$$\operatorname{tr}_2(r_{12}A_2),\ A_2 = 1 \otimes A \tag{21}$$

and we have

$$r_{12}A_2 = \sum_{ij,kl} r_{ij,kl} E_{ij} \otimes (E_{kl}A). \tag{22}$$

The trace of the second component has the form

$$\operatorname{tr}_2(r_{12}A_2) = \sum_{ij,kl} r_{ij,kl} E_{ij} A_{lk} r_{12} = \sum_{ij,kl} r_{ij,kl} E_{ij} \otimes E_{kl}. \tag{23}$$

Linear Poisson-Lie structure

Lie algebra is a vector space equipped with the commutator, which is rather similar to the Poisson bracket. For generators T_α, T_β of the Lie algebra $\mathfrak{g}$ we have

$$[T_\alpha, T_\beta] = \sum_\gamma C_{\alpha\beta}^\gamma T_\gamma, \tag{24}$$

where $C^{\gamma}_{\alpha\beta}$ are called structure constants. The commutator also satisfies Jacobi identity. On Lie coalgebra $\mathfrak{g}^*$ we have the same identity for the generators

$$\{x_\alpha, x_\beta\} = \sum_\gamma C^{\gamma}_{\alpha\beta} x_\gamma. \tag{25}$$

In case of $\mathfrak{su}(2)$ for Pauli matrices we have

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \tag{26}$$

$$[\sigma_2, \sigma_3] = 2i\sigma_1, \tag{27}$$

$$[\sigma_3, \sigma_1] = 2i\sigma_2. \tag{28}$$

Generally, $[\sigma_\alpha, \sigma_\beta] = 2i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma$. In spin 1/2 representation Pauli matrices have the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{29}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{30}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{31}$$

Poisson brackets for our case have the form

$$\{x_1, x_2\} = x_3, \tag{32}$$

$$\{x_2, x_3\} = x_1, \tag{33}$$

$$\{x_3, x_1\} = x_2. \tag{34}$$

Such Poisson brackets arise when we deal with the Euler system, defined by

$$H = \frac{1}{2}\left(J_1 x_1^2 + J_2 x_2^2 + J_3 x_3^2\right). \tag{35}$$

It describes the rotation of a solid body in 3-dimensional space.

In case of $\mathfrak{gl}_N$, which contains $N \times N$ matrices we have

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}, \tag{36}$$

which can be easily verified taking into account the definition of E_{ij}. Poisson structure is defined by

$$S = \sum_{ij} S_{ij} E_{ij}, \tag{37}$$

$$\{S_{ij}, S_{kl}\} = \delta_{kj} S_{il} - \delta_{il} S_{kj}.$$

Lecture 2

Permutation operator

Last time we dealt with the Poisson linear structure

$$\{S_{ij}, S_{kl}\} = \delta_{il} S_{kj} - \delta_{kj} S_{il}. \tag{38}$$

It can be rewritten as

$$\{S_1, S_2\} = [P_{12}, S_1],$$

where P_{12} is a matrix permutation operator. It acts on a tensor product of 2 matrices

$$P_{12} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}, \tag{39}$$

where E_{ij} is the $N \times N$ matrix, where 1 stands in the i-th row and j-th column. We also use the following notation $(E_{ij})_{ab} = \delta_{ia}\delta_{jl}$. For $u, v \in \mathbb{C}^n$ we have

$$P_{12}(u \otimes v) = v \otimes u.$$

In order to verify this, one has to introduce the standard basis $e_i = (0, 0 \ldots, 1, 0, \ldots, 0)$ and use the multiplication rule

$$E_{ik} e_k = \delta_{jk} e_i.$$

Let us now verify (39). Consider

$$\left(\sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}\right) \left(\sum_{k} u_k e_k \otimes \sum_{m} v_m e_m\right) = \sum_{ijkm} u_k v_m (E_{ij} e_k \otimes E_{ji} e_m) = \sum_{ijkm} u_k v_m \delta_{jk} \delta_{im} e_i \otimes e_j.$$

The next property is that for $A, B \in \mathrm{Mat}_N$ we have

$$P_{12}(A \otimes B) = (B \otimes A) P_{12}. \tag{40}$$

To prove it, consider

$$\left(\sum_{i,j} E_{ij} \otimes E_{ji}\right) \left(\sum_{a,b} E_{ab} \otimes A_{ab}\right) \otimes \left(\sum_{a,b} E_{cd} \otimes B_{cd}\right) = \sum_{ijabcd} (E_{ij} E_{ab}) \otimes (E_{ji} E_{cd}) A_{ab} B_{cd} = \tag{41}$$

$$= \sum_{ijabcd} E_{ib} \delta_{aj} \otimes E_{jd} \delta_{il} A_{ab} B_{cd} = \sum_{abcd} E_{cb} \otimes E_{ab} A_{ab} B_{cd}. \tag{42}$$

The next property is that $P_{12}^2 = 1_N \otimes 1_N = 1_{N^2}$. Let us do it explicitly:

$$\left(\sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}\right) \left(\sum_{k,l=1}^{N} E_{kl} \otimes E_{lk}\right) = \sum_{i,j,k,l=1}^{N} (E_{ij} E_{kl}) \otimes (E_{ji \otimes E_{kl}}) = \sum_{i,j,k,l=1}^{N} \delta_{kj} \delta_{i;} E_{il} \otimes E_{jj} = \tag{43}$$

$$= \sum_{i,j} E_{ii} \otimes E_{jj} = \left(\sum_i E_{ii}\right) \otimes \left(\sum_j E_{jj}\right). \tag{44}$$

In case $N = 2$ we have

$$P_{12} = E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{22} \otimes E_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{45}$$

The next property is that

$$\operatorname{tr}_2(P_{12} A_2) = A, \tag{46}$$

where $A_2 = 1 \otimes A$. To verify this, consider

$$\left(\sum_{i,j} E_{ij} \otimes E_{ji}\right)(1_N \otimes A) = \sum_{i,j} E_{ij} \otimes E_{ji} A \tag{47}$$

and by taking trace we obtain

$$\sum_{i,j} E_{ij} \otimes \operatorname{tr}(E_{ji} A) = \sum_{k,l} \operatorname{tr}(E_{ij} E_{kl}) A_{kl} = \sum_{k,l} \delta_{kj} \operatorname{tr}(E_{il}) A_{kl} = A_{ji}. \tag{48}$$

The permutation operator may also act on more than 2 vector spaces. In detail, for $a, b, c \in V \simeq \mathbb{C}^N$

$$P_{12}(a \otimes b \otimes c) = b \otimes a \otimes c, \tag{49}$$

$$P_{12} = \sum_{i,j} E_{ij} \otimes E_{ji} \otimes 1_N. \tag{50}$$

The same can be written for P_{13}:

$$P_{13} = \sum_{i,j} E_{ij} \otimes 1_N \otimes E_{ji}. \tag{51}$$

When components are different, operators commute

$$P_{12} P_{34} = P_{34} P_{12}. \tag{52}$$

In particular,

$$P_{12} P_{23} = P_{13} P_{12} = P_{23} P_{13}. \tag{53}$$

For example,

$$P_{13} P_{12}(a \otimes b \otimes c) = P_{13}(b \otimes a \otimes c) = c \otimes a \otimes b = A_{ji} \tag{54}$$

and

$$P_{13} P_{12}(a \otimes b \otimes c) = P_{13}(a \otimes c \otimes b) = c \otimes a \otimes b = A_{ji}. \tag{55}$$

Let us now discuss some applications of the permutation operator. From the previous lecture we have

$$S = \sum_{ij} S_{ij} E_{ij}, \tag{56}$$

$$\{S_{ij}, S_{kl}\} = \delta_{kj} S_{il} - \delta_{il} S_{kj}.$$

Then we can write

$$\{S_1, S_2\} = [P_{12}, S_1] = P_{12}S_1 - S_1 P_{12}, \tag{57}$$

where

$$P_{12}S_1 = \left(\sum_{a,b} E_{ab} \otimes E_{ba}\right)\left(\sum_{c,d} E_{cd} \otimes S_{cd} \otimes 1\right) = \sum_{a,b,c,d} S_{cd}(E_{ab}E_{cd}) \otimes E_{ba} = \tag{58}$$

$$= \sum_{a,b,c,d} S_{cd}\delta_{bc} E_{ad} \otimes E_{ba} = \sum_{a,b,d} S_{bd} E_{ad} \otimes E_{ba} = \sum E_{ij} \otimes E_{kl} S_{kj} \delta_{il}. \tag{59}$$

The second terms is for homework.

Spectral parameter

Consdier the Lax eequation

$$\dot{L} = [L, M], \tag{60}$$

where $L, M \in \mathrm{Mat}_N$. We also have conserved quantities

$$H_k = \frac{1}{k}\,\mathrm{tr}(L^k). \tag{61}$$

Consider the eigenvalue problem

$$L\Psi = \Psi\Lambda, \tag{62}$$

from which we obtain

$$L = \Psi\Lambda\Psi^{-1}. \tag{63}$$

Using the latter equation one can easily compute powers of L:

$$L^k = \Psi\Lambda^k\Psi^{-1}. \tag{64}$$

Trace is also easily computed

$$\mathrm{tr}(L^k) = \mathrm{tr}(\Lambda^k) = \sum_{i=1}^{N} \lambda_i \tag{65}$$

Consider now $L(z)$ — a matrix-valued function of z. The Lax equation is written in the form

$$\dot{L}(z) = [L(z), M(z)]\ \forall z. \tag{66}$$

For example,

$$L(z) = \frac{A}{z} + B \tag{67}$$

and

$$\dot{L}(z) = \frac{\dot{A}}{z} + \dot{B} = \frac{[A, C]}{z} + [B, C]. \tag{68}$$

This equation is equivalent to 2 equations:

$$\dot{A} = [A, C], \tag{69}$$

$$\dot{B} = [B, C]. \tag{70}$$

By using z (spectral parameter) we cam rewrite

$$\{L_1, L_2\} = [L_1, r_{12}] - [L_2, r_{21}] \tag{71}$$

in the following way

$$\{L_1(z), L_2(z)\} = [L_1(z), r_{12}(z, w)] - [r_{21}(w, z), L_2(w)]. \tag{72}$$

If we consider

$$L = \frac{A}{z} + B, \tag{73}$$

for the trace we have

$$\operatorname{tr} L^2)(z) = \frac{\operatorname{tr}(\mathrm{A}^2)}{z^2} + \frac{2\operatorname{tr}(AB)}{z} + \operatorname{tr}(B^2). \tag{74}$$

All coefficients, corresponding to trace, are conserved quantities. By considering

$$\frac{d}{dt} L^k(z) = \sum_m z^m \dot{H}_{m,k} = 0 \tag{75}$$

we have a large number of conserved quantities (integrals of motion).

Let us analyse some examples. Consider

$$r_{12}(z, w) = \frac{P_{12}}{z - w}, \tag{76}$$

where $\operatorname{Res}_{z=w} r_{12}(z, w) = P_{12}$. Define $L(z) = \frac{S}{z}$. Then

$$\operatorname{tr} L^k(z) = \frac{\operatorname{tr}(\mathrm{S}^\mathrm{k})}{z^k}. \tag{77}$$

The bracket will have the form (the left-hand side

$$\{L_1(z), L_2(w)\} = \{\frac{S_1}{z}, \frac{S_2}{w}\} = \frac{1}{zw}\{S_1, S_2\}, \tag{78}$$

whilst the right hand side will be

$$\left[\frac{S_1}{z}, \frac{P_{12}}{z-w}\right] - \left[\frac{S_2}{w}, \frac{P_{12}}{w-z}\right] = \frac{1}{z-w}\frac{1}{z}[S_1, P_{12}] + \frac{1}{(z-w)w}[S_2, P_{12}]. \tag{79}$$

Since

$$[S_1, P_{12}] = S_1 P_{12} - P_{12} S_1 = P_{12} S_2 - S_2 P_{12} = -[S_2, P_{12}] \tag{80}$$

we have

$$\frac{1}{(z-w)w}[S_2, P_{12}] = [S_1, P_{12}]\left(\frac{1}{z-w}\frac{1}{z} - \frac{1}{z-w}\frac{1}{w}\right) = -\frac{1}{zw}[S_1, P_{12}] = \frac{1}{zw}[P_{12}, S_1]. \tag{81}$$

Quadratic r-matrix structure

Consider

$$\{L_1(z), L_2(w)\} = [L_1(z)L_2(w), r_{12}(z-w)], \tag{82}$$

where $L_1(z)L_2(w) = L(z) \otimes L(w)$. Then

$$\{\operatorname{tr} L^k(z), \operatorname{tr} L^m(w)\} = 0. \tag{83}$$

For $L(z) = 1 + \frac{S}{z}$ one can compute

$$\{L_1(z), L_2(w)\} = -\left[\frac{P_{12}}{z-w}, \left(1+\frac{S_1}{z}\right)\left(1+\frac{S_2}{w}\right)\right] = -\left[\frac{P_{12}}{z-w}, \frac{S_1}{z}+\frac{S_2}{w}\right]. \tag{84}$$

Proposition 2. *Suppose we have a set of* $\{L^i(z)\}_{i=1}^n$ *and*

$$\{L_1^i(z)L_2^j(w)\} = \delta^{ij}[L_1^i(z)L_2^i(w), r_{12}(z-w)].$$

Then

$$T(z) = L^1(z)L^2(z)\dots L^n(z) \tag{85}$$

also satisfies the quadratic r-matrix structure identity

$$\{T_1(z), T_2(w)\} = [T_1(z)T_2(w), r_{12}(z-w)]. \tag{86}$$

Here $T(z)$ *is a monodromy matrix.*

Lecture 3

Last time we considered the quadratic $r-$matrix structure

$$\{L_1(z), L_2(w)\} = [r_{12}(z, w), L_1(z)L_2(w)], \tag{87}$$

where $L_1(z)L_2(w) := L(z) \otimes L(w)$. It has the following properties

1) $\{\operatorname{tr} L^k(z), \operatorname{tr} L^m(w)\} = 0$,

2) $\{\{L_1, L_2\}, L_3\} + cycl = 0$, which provides the classical Yang-Baxter equation.

3) Suppose there is a set of quadratic brackets

$$\{L_1^i(z), L_2^j(w)\} = \delta^{ij}[r_{12}(z, w), L_1^i(z)L_2^j(w)]. \tag{88}$$

Then the monodromy matrix

$$T(z) = L^1(z)L^2(z) \dots L^n(z) \tag{89}$$

also satisfies the quadratic relation

$$\{T_1(z)T_2(w)\} = [r_{12}(z, w), T_1(z)T_2(w)]. \tag{90}$$

The monodromy matrix is significant when we consider the eigenvalue problem

$$L^1\Psi^1 = \lambda\Psi_2, \tag{91}$$

$$\Psi^3 = L^2\Psi^2 = L^2L^1\Psi^1 \tag{92}$$

and so on. Speaking non-formally, monodromy matrix bridges the connection between the solutions of eigenvalue problems. It is also significant since the Lax pair

$$\dot{L} = [L, M] \tag{93}$$

or

$$[L, \partial_t + M] = 0 \tag{94}$$

is a compatibility condition for the following system

$$\begin{cases} (\partial_t + M)\Psi = 0, \\ L\Psi = \lambda\Psi. \end{cases} \tag{95}$$

Consider an example: some double-sided model with A, B — matrices, which constitute some integrable system on the one and another side correspondingly. Then for $A_1 = 1_N \otimes A$ and $A_2 = 1_n \otimes A$ we have

$$\begin{cases} \{A_1, A_2\} = [r_{12}, A_1A_2], \\ \{B_1, B_2\} = [r_{12}, B_1B_2], \\ \{A_1, B_2\} = 0. \end{cases} \tag{96}$$

We want to show that $T_1 = A_1B_1 = (A \otimes 1_N)(B \otimes 1_N) = T \otimes 1_N$ also satisfies the same quadratic relation.
Consider $r_{12}(z,w) = \frac{P_{12}}{z-w}$ and $L(z) = 1_N + \frac{S}{z}$, then

$$[r_k, L_1^1 L_2^2] = \left[\frac{P_{12}}{z-w}, \left(1 + \frac{S_1^1}{z}\right)\left(1 + \frac{S_2^1}{z}\right)\right] = \frac{S_1}{z} + \frac{S_2}{w} \tag{97}$$

taking into account

$$[P_{12}, S \otimes S] = P_{12}S \otimes S - S \otimes SP_{12} = 0. \tag{98}$$

Let us consider

$$\operatorname{tr} L_1^2(z) = \operatorname{tr}\left(1 + \frac{2S^1}{z} + \frac{(S^1)^2}{z^2}\right) \tag{99}$$

and we notice that the coefficients inside the trace are the Casimir functions and they provide no non-trivial dynamics.
Consider now (with the help of Leibniz rule)

$$\{T_1, T_2\} = \{A_1B_1, A_2B_2\} = A_1\{B_1, A_2B_2\} + \{A_1, A_2B_2\}B_1 = \tag{100}$$

$$= A_1\{B_1, A_2\}B_2 + A_1A_2\{B_1, B_2\} + \{A_1, A_2\}B_2B_1 + A_2\{A_1, B_2\}B_1. \tag{101}$$

and since A, B commute, some brackets vanish and we have

$$\{T_1, T_2\} = A_1A_2(r_{12}B_1B_2 - B_1B_2r_{12}) + (r_{12}A_1A_2 - A_1A_2r_{12})B_1B_2 = \tag{102}$$

$$= A_1A_2r_{12}B_1B_2 - A_1A_2B_1B_2r_{12} + r_{12}A_1A_2B_1B_2 - A_1A_2r_{12}B_1B_2. \tag{103}$$

Since

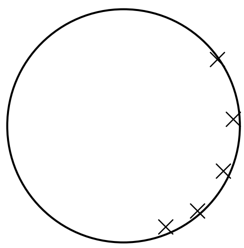
$$[A_1, B_2] = 0, \tag{104}$$

we obtain that

$$\{T_1, T_2\} = [r_{12}, T_1, T_2]. \tag{105}$$

Quantisation

The idea of quantisation is that we replace the functions with the operators $f \to \hat{f}$ acting on some Hilbert space $\mathcal{H}$ of states.
Consider the classical chain model



satisfying

$$\{S_{ab}, S_{cd}\} = \delta_{ad} S^i_{cb} - \delta_{cd} S^i_{ad} \tag{106}$$

and

$$\{S^i_{ab}, S^i_{cd}\} = 0. \tag{107}$$

We wish to quantise these Poisson brackets. Remember that the connection between the classical and quantum cases is described by (**??**). This bracket arose from Lie algebra $\mathfrak{gl}_N$

$$[E_{ab}, E_{cd}] = E_{ad}\delta_{bc} - E_{cb}\delta_{ad}. \tag{108}$$

In order to quantise the bracket, define on $\mathcal{H} := V \otimes V \otimes \cdots \otimes V := V^{\otimes n}$ operators $\hat{S}^i_{ab} := \hbar E^i_{ba}$

$$\hat{S}^i_{ab} = 1_N \otimes 1_N \cdots \otimes \hbar E_{ba} \otimes \cdots \otimes 1_N \in \mathrm{End}(\mathcal{H}) \tag{109}$$

and now the following identity holds

$$[\hat{S}^i_{ab}, \hat{S}^j_{cd}] = 0. \tag{110}$$

Consider now

$$[\hat{S}^i_{ab}, \hat{S}^i_{cd}] = \hbar^2 [E^i_{ba}, E^i_{dc}] = \hbar^2 E^i_{bc}\delta_{ad} - \hbar^2 E^i_{da}\delta_{bc} = \hbar \hat{S}^i_{cb}\delta_{ad} - \hbar \hat{S}^i_{ad}\delta_{bc} \tag{111}$$

and wee see that in terms of the limit

$$\{S^i_{ab}, S^i_{cd}\} = \lim_{\hbar \to 0} \frac{[\ldots]}{\hbar} \tag{112}$$

everything is correct.

Let us now quantise the Lax matrix $L(z) \to \hat{L}(z) = 1 + \frac{\hat{S}}{z}$, where

$$\hat{S} = \sum E_{ij} \hat{S}_{ij}. \tag{113}$$

Consider the quantum R-matrix

$$R^\hbar_{12}(z, w) = \sum_{i,j,k,l=1}^{N} E_{ij} \otimes E_{kl} R_{ijkl}(\hbar, z, w), \tag{114}$$

with the help of which one can write the quantum Yang-Baxter equation

$$R^\hbar_{12}(z_1, z_2) R^\hbar_{13}(z_1, z_3) R^\hbar_{23}(z_2, z_3) = R^\hbar_{23}(z_2, z_3) R^\hbar_{13}(z_1, z_3) R^\hbar_{12}(z_1, z_2). \tag{115}$$

This equation is written in $\mathrm{Mat}_N^{\otimes 3}$ and it is written up to a normalisation, since the function is cancelled out, when the equation is multiplied by it.

Proposition 3. *The Yang's R-matrix*

$$R^\hbar_{12}(z, w) = 1 \otimes 1 + \hbar \frac{P_{12}}{z - w} \tag{116}$$

satisfies the quantum Yang-Baxter equation.

The Lax matrix has the form

$$\hat{L}(z) = 1 + \frac{\hat{S}}{z} = 1 + \frac{\sum_{i,j=1}^{N} E_{ij}\hat{S}_{ij}}{z} \tag{117}$$

and we have the important identity

$$R_{12}^{\hbar}(z,w) = 1 \otimes 1 + \hbar r_{12} + O(\hbar^2). \tag{118}$$

By using the latter expansion we can compare the coefficients in front of $\hbar$:

$$\hbar^0 : 1 = 1, \tag{119}$$

$$\hbar^1 : r_{12} + r_{13} + r_{23} = r_{12} + r_{13} + r_{23}, \tag{120}$$

$$\hbar^2 : r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23} = r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12}. \tag{121}$$

The last equation has the form

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}.r_{23}] = 0, \tag{122}$$

which is exactly the classical Yang-Baxter equation.

Let us get back to the quantisation

$$\hat{L}^i(z) = 1 + \frac{\hat{S}^i}{z} = 1 + \frac{\sum_{a,b=1}^{N} E_{a,b}\hat{S}_{ab}}{z} = 1 + \frac{\hbar\sum_{a,b=1}^{N} E_{a,b}E_{ba}^i}{z} = R_{oi}^{\hbar}(z). \tag{123}$$

The left-hand side has some matrix structure and belongs to so called auxillary space, whilst the right-hand side contains the permutation operator. So, we can write it as

$$\hat{L}_0^i(z) = R_{oi}^{\hbar}(z) \tag{124}$$

Let us now introduce the notation $1 \to 0, 2 \to 0', 3 \to i$, then the quadratic R-matrix structure has the form

$$\{L_0(z), L_{0'}(w)\} = [r_{00'}(z,w), L_0(z)L_{0'}(w)]. \tag{125}$$

Here $0, 0'$ belong to auxillary spaces. The components belonging to the Hilbert spaces are denoted $1, \ldots, n$. The quantum Yang-Baxter equation has the following form

$$R_{00'}^{\hbar}(z-w)R_{0i}^{\hbar}(z)R_{0'i}^{\hbar}(w) = R_{0'i}^{\hbar}(w)R_{0i}^{\hbar}(z)R_{00'}^{\hbar}(z-w). \tag{126}$$

Finally, let us rewrite this equation as so called RLL or RTT, or exchange relation

$$R_{00'}^{\hbar}(z-w)\hat{L}_0(z)\hat{L}_{0'}(w) = \hat{L}_{0'}(w)\hat{L}_0(z)R_{00'}^{\hbar}(z-w). \tag{127}$$

By inserting there our expansion with respect to $\hbar$ we obtain

$$(1 + \hbar r_{00'})\hat{L}_0(z)\hat{L}_{0'}(w) = (1 + \hbar r_{00'})\hat{L}_{0'}(w)\hat{L}_0(z). \tag{128}$$

By comparing coefficients and dividing by $\hbar$ we get

$$\frac{\hat{L}_0(z)\hat{L}_{0'}(w) - \hat{L}_{0'}(w)\hat{L}_0(z)}{\hbar} = (\hat{L}_{0'}\hat{L}_0 r_{00'} - r_{00'}\hat{L}_0\hat{L}_{0'}). \tag{129}$$

It is important to mention that in quantum mechanics operators do not generally commute

$$\hat{A}_1\hat{B}_2 = (\hat{A}\otimes 1)(1\otimes\hat{B}) = (\sum_{ijkl} E_{ij}\hat{A}_{ij}\otimes 1)(1\otimes E_{kl}\hat{B}_{kl}) = \sum_{ijkl} E_{ij}\otimes E_{kl}\hat{A}_{ij}\hat{B}_{kl}, \tag{130}$$

but in different order we get

$$\hat{B}_2\hat{A}_1 = \sum_{ijkl} E_{ij}\otimes E_{kl}\hat{B}_{kl}\hat{A}_{ij}. \tag{131}$$

In terms of R-matrix

$$R_{12}\hat{L}_1\hat{L}_2 = \hat{L}_2\hat{L}_1 R_{12}, \tag{132}$$

which is equivalent to

$$R_{12}\hat{L}_1\hat{L}_2 R_{12}^{-1} = \hat{L}_2\hat{L}_1 \tag{133}$$

what underlines non-commutativity once again.

Lecture 4

Last time we discussed the quantisation procedure. To be precise, we derived

$$R^{\hbar}_{00'}(z-w)\hat{L}_0(z)\hat{L}_{0'}(w) = \hat{L}_{0'}(w)\hat{L}_0(z)R^{\hbar}_{00'}(z-w). \tag{134}$$

This relation satisfies the Yang-Baxter equation

$$R^{\hbar}_{12}(z_1-z_2)R^{\hbar}_{13}(z_1-z_3)R^{\hbar}_{23}(z_2-z_3) = R^{\hbar}_{23}(z_2-z_3)R^{\hbar}_{13}(z_1-z_3)R^{\hbar}_{12}(z_1-z_2). \tag{135}$$

By considering the expansion

$$R^{\hbar}_{12}(z_1-z_2) = 1\otimes 1 + \hbar r_{12}(z_1-z_2) + O(\hbar^2) \tag{136}$$

we can get back to the classical Yang-Baxter equation, which was initially derived from the Jacobi identity. We can also interpret it as a compatibility condition. First, consider

$$\hat{L}_1\hat{L}_2\hat{L}_3 \to^{R_{23}} \hat{L}_1\hat{L}_3\hat{L}_2 \to^{R_{13}} \hat{L}_3\hat{L}_1\hat{L}_2 \to^{R_{12}} \hat{L}_3\hat{L}_2\hat{L}_1, \tag{137}$$

where $\hat{L}_1\hat{L}_2\hat{L}_3$ is affected by the right-hand side of the Yang-Baxter equation

$$R_{23}R_{13}R_{12}\hat{L}_1\hat{L}_2\hat{L}_3 = R_{23}R_{13}\hat{L}_2\hat{L}_1R_2\hat{L}_3 = R_{23}R_{13}\hat{L}_2\hat{L}_1\hat{L}_3R_{12} = R_{23}\hat{L}_2R_{13}\hat{L}_1\hat{L}_3R_{12} = \tag{138}$$

$$= R_{23}\hat{L}_2\hat{L}_3\hat{L}_1R_{13}R_{12} = \hat{L}_3\hat{L}_2R_{23}\hat{L}_1R_{13}R_{12} = \hat{L}_3\hat{L}_2\hat{L}_1R_{23}R_{13}R_{12}. \tag{139}$$

Finally, we see that

$$\hat{L}_1\hat{L}_2\hat{L}_3 = rhs^{-1}\hat{L}_3\hat{L}_2\hat{L}_1 rhs, \tag{140}$$

where rhs stands for the 'right-hand side'. In the similar way one can obtain the expression for $R_{12}R_{13}R_{23}\hat{L}_1\hat{L}_2\hat{L}_3$ and

$$\hat{L}_1\hat{L}_2\hat{L}_3 = lhs^{-1}\hat{L}_3\hat{L}_2\hat{L}_1 lhs, \tag{141}$$

where lhs stands for the 'left-hand side'.

Consider

$$\{L_0(z), L_{0'}(w)\} = [r_{00'}(z-w), L_0(z)L_{0'}(w)] \tag{142}$$

and as an example, Poisson brackets on the Lie group SL_2

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{143}$$

where $ad-bc=1$. Then the following identity holds

$$\{T_1, T_2\} = [r_{12}, T_1T_2]. \tag{144}$$

Here

$$r_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{145}$$

or, in other words,

$$r_{12} = E_{12} \otimes E_{21} - E_{21} \otimes E_{12}. \tag{146}$$

Let us write now the set of brackets

$$\begin{cases} \{a,b\} = ab, \\ \{b,c\} = 0, \\ \{b,d\} = bd, \\ \{a,c\} = ac, \\ \{c,d\} = cd, \\ \{a,d\} = 2bc, \end{cases} \tag{147}$$

which actually define the Poisson-Lie structure. This example corresponds to the relativistic Toda chain model. Consider the variables

$$a = e^p\sqrt{1+e^{2q}}, \tag{148}$$

$$b = e^q = c, \tag{149}$$

$$d = e^{-p}\sqrt{1+e^{2q}}, \tag{150}$$

$$\{p,q\} = 1 \tag{151}$$

and let us quantise this model. The Quantum R-matrix was suggested by Infeld and it has the form

$$R_{12}^{\hbar} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{152}$$

where $q = e^{i\hbar}$. Then we can introduce the quantum T-matrix

$$\hat{T} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}. \tag{153}$$

We simply introduced new operators acting on some Hilbert space. The operators satisfy the commutation relations

$$\begin{cases} \hat{B}\hat{C} = \hat{C}\hat{B}, \\ \hat{A}\hat{B} = q\hat{B}\hat{A}, \\ \hat{A}\hat{C} = q\hat{C}\hat{A} \\ \hat{B}\hat{D} = q\hat{D}\hat{B}, \\ \hat{C}\hat{D} = q\hat{D}\hat{C}, \\ \hat{A}\hat{D} - \hat{D}\hat{A} = (q-q^{-1})\hat{B}\hat{C}. \end{cases} \tag{154}$$

This set of relations defines the quantum algebra. The operators can be explicitly defined as

$$\hat{A} = \sqrt{1+e^{2q+i\hbar}}e^{\hat{p}}, \tag{155}$$

$$\hat{B} = \hat{C} = e^q, \tag{156}$$

$$\hat{D} = e^{-\hat{p}}\sqrt{1 + e^{2q+i\hbar}}, \tag{157}$$

where

$$e^{\hat{p}}e^{\hat{q}} = e^{i\hbar}e^{\hat{q}}e^{\hat{p}}. \tag{158}$$

Let us get back to integrability. If we have the set of quadratic Poisson brackets satisfying the identities

$$\{L_0^i, L_{0'}^i\} = [r_{00'}, L_0^i, L_{0'}^i],\ i = 1, \dots, n$$

and

$$\{L_0^i, L_{0'}^j\} = 0,\ i \neq j. \tag{159}$$

Then for T_1 and T_2 we have

$$\{T_1, T_2\} = [r_{12}, T_1 T_2]. \tag{160}$$

Let us now consider the set of relations

$$\begin{cases} R^{\hbar}_{00'}\hat{L}_0^i\hat{L}_{0'}^i = \hat{L}_{0'}^i\hat{L}_0^i R^{\hbar}_{00'}, \\ \hat{L}_0^i\hat{L}_{0'}^j = \hat{L}_{0'}^j\hat{L}_0^i = 0,\ i \neq j. \end{cases} \tag{161}$$

The quantum monodromy matrix is defined as $\hat{T} := \hat{L}^n \dots \hat{L}^1$ and we claim that it satisfies the quadratic relation

$$R^{\hbar}_{00'}\hat{T}_0\hat{T}_{0'} = \hat{T}_{0'}\hat{T}_0 R^{\hbar}_{00'}.$$

We can insert there the spectral parameter and additionally supposing that R-matrix is invertible we can obtain

$$R^{\hbar}_{00'}(z-w)\hat{T}_0(z)\hat{T}_{0'}(w) = \hat{T}_{0'}(w)\hat{T}_0(z)R^{\hbar}_{00'}(z-w) \tag{162}$$

and, hence,

$$\hat{T}_0(z)\hat{T}_{0'}(w) = (R^{\hbar}_{00'}(z-w))^{-1}\hat{T}_{0'}(w)\hat{T}_0(z)R^{\hbar}_{00'}(z-w). \tag{163}$$

Taking trace from both componetns we get

$$\mathrm{tr}_0(\hat{T}_0(z)) \cdot \mathrm{tr}_{0'}(\hat{T}_{0'}(w)) = \mathrm{tr}_{0'}(\hat{T}_{0'}(w))\,\mathrm{tr}_0(\hat{T}_0(z)). \tag{164}$$

The trace of monodromy matrix

$$\hat{t}(z) := \mathrm{Tr}_0(\hat{T}(z)). \tag{165}$$

For the trace we have

$$\hat{t}(z)\hat{t}(w) = \hat{t}(w)\hat{t}(z), \tag{166}$$

where

$$\hat{t}(z) = \sum_k z^k \hat{H}_k, \tag{167}$$

which actually lies within the concept of quantum integrability

$$[\hat{H}_k, \hat{H}_m] = 0. \tag{168}$$

Thus, we can define the Schroedinger equation $\forall k$

$$\hat{H}_k \Psi = \lambda_k \Psi. \tag{169}$$

If $\hat{T} := \hat{L}_0^i \hat{L}_0^{i+1}$ then we have

$$R_{00'} \hat{T}_0 \hat{T}_{0'} = R_{00'} \hat{L}_0^i \hat{L}_0^{i+1} \hat{L}_{0'}^i \hat{L}_{0'}^{i+1} \tag{170}$$

and by permuting the needed terms we get

$$R_{00'} \hat{T}_0 \hat{T}_{0'} = R_{00'} \hat{L}_0^i \hat{L}_{0'}^i \hat{L}_0^{i+1} \hat{L}_{0'}^{i+1} = \hat{L}_{0'}^i \hat{L}_0^i R_{00'} \hat{L}_0^{i+1} \hat{L}_{0'}^{i+1} = \hat{L}_{0'}^i \hat{L}_0^i \hat{L}_0^{i+1} \hat{L}_{0'}^{i+1} R_{00'} = \hat{T}_{0'} \hat{T}_0 R_{00'}. \tag{171}$$

Let us now construct the quantum spin chain. Consider

$$R_{00'}^{\hbar} = 1 \otimes 1 + \frac{\hbar}{z} P_{00'} \tag{172}$$

or, in matrix form,

$$R_{00'}^{\hbar} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{173}$$

Define now

$$\hat{L}_0^i(z) = 1 + \frac{1}{z} \hat{S}. \tag{174}$$

where

$$\hat{S} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix} \tag{175}$$

and $\hat{S}_{ij} = \hbar E_{ji}$. So,

$$\hat{L}_0^i(z) = R_{0i}^{\hbar}(z). \tag{176}$$

The for the monodromy matrix we have

$$\hat{T}_0 = \hat{L}_0^n \dots \hat{L}_0^1 = R_{0,n}^{\hbar}(z) R_{0,n-1}^{\hbar}(z) \dots R_{0,1}^{\hbar}(z) = \begin{pmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{pmatrix}. \tag{177}$$

Operators in the latter equation act on End $(\mathcal{H})$, where $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ We can now compute the trace and we obtain

$$\operatorname{tr}_0 \hat{T}_0(z) = \hat{A}(z) + \hat{B}(z). \tag{178}$$

For $\hat{S}$ we have

$$\hat{S} = \sum_{a,b} E_{ab}\hat{S}^i_{ab} \tag{179}$$

and by quantising

$$\rho(\hat{S}) = \hbar \sum_{a,b} E_{ab}E_{ba} = \hbar P_{0i}. \tag{180}$$

Our main purpose is to diagonalise $2^n \times 2^n$ matrix, its dimension can be enormous.

Let us now introduce the notation for the next lecture; we can redefine $\hat{L}^i(z) \to \hat{L}^i(z-z_1)$, so we obtain so called homogeneous parameter definition. Then let us introduce

$$\hat{L}^i(z) = \begin{pmatrix} 1 + \frac{\hbar}{z-z_1}E^i_{11} & \frac{\hbar}{z-z_1}E^i_{21} \\ \frac{\hbar}{z-z_1}E^i_{11} & 1 + \frac{\hbar}{z-z_1}E^i_{22} \end{pmatrix} \tag{181}$$

or, alternatively,

$$\hat{L}^i(z) = \begin{pmatrix} \hat{\alpha}^i(z) & \hat{\beta}^i(z) \\ \hat{\gamma}^i(z) & \hat{\delta}^i(z) \end{pmatrix}. \tag{182}$$

Consider now the Schroedinger equation for the trace of the monodromy matrix

$$\hat{t}(z)|\Psi(z)\rangle = t(z)|\Psi\rangle, \tag{183}$$

where Ψ does not depend on z since

$$[\hat{t}(z), \hat{t}(w)] = 0. \tag{184}$$

The same equation can be written for w and so on. Our purpose is to solve this equation for the spin chain. We can obtain a local Hamiltonian defined by

$$\hat{H}_{local} = \sum_{i=1}^{n} P_{i,i+1}. \tag{185}$$

Define vacuum vector ($e_1 = (1,0)^T$)

$$|0\rangle := e_1 \otimes e_1 \otimes \cdots \otimes e_1, \tag{186}$$

hence we know how, for instance, E^i_{21} acts on vacuum vector. Next time we are going to define the action of operators $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ on $|0\rangle$.

Lecture 5

Bethe Ansatz method

Consider the monodromy matrix

$$\hat{T}_0(z) = \hat{L}_0^n(z - z_n)\hat{L}_0^{n-1}(z - z_{n-1}) \dots \hat{L}_0^1(z - z_1) \tag{187}$$

or,equivalently,

$$\hat{T}_0 = \hat{L}_0^n \dots \hat{L}_0^1 = R_{0,n}^{\hbar}(z) R_{0,n-1}^{\hbar}(z) \dots R_{0,1}^{\hbar}(z) = \begin{pmatrix} \hat{A}(z) & \hat{B}(z) \\ \hat{C}(z) & \hat{D}(z) \end{pmatrix}. \tag{188}$$

acting on the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$. Here

$$\hat{L}^i(z) = \begin{pmatrix} 1 + \frac{\hbar}{z-z_1} E_{11}^i & \frac{\hbar}{z-z_1} E_{21}^i \\ \frac{\hbar}{z-z_1} E_{11}^i & 1 + \frac{\hbar}{z-z_1} E_{22}^i \end{pmatrix}. \tag{189}$$

We can also use (as we already know) different notation

$$\hat{L}^i(z) = \begin{pmatrix} \hat{\alpha}^i(z) & \hat{\beta}^i(z) \\ \hat{\gamma}^i(z) & \hat{\delta}^i(z) \end{pmatrix}. \tag{190}$$

Then we know that by denoting $\hat{t}(z) = \operatorname{tr}_0 T_0(z) = \hat{A}(z) + \hat{B}(z)$ we have the following relation

$$[\hat{t}(z), \hat{t}(w)] = 0. \tag{191}$$

Our goal is to solve the Schroedinger equation

$$\hat{t}(z)|\Psi\rangle = \tau(z)|\Psi\rangle. \tag{192}$$

The vacuum vector has the form

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{193}$$

Let us consider the action of our operators:

$$\hat{\alpha}_i(z)|0\rangle = \left(1 + \frac{\hbar}{z - z_i}\right)|0\rangle \tag{194}$$

and in the same manner

$$\hat{\delta}_i(z)|0\rangle = \delta_i(z)|0\rangle \tag{195}$$

$$\hat{\gamma}_i(z)|0\rangle = 0. \tag{196}$$

We also keep in mind that

$$E_{11}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$E_{12}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{197}$$

Theorem 2. *The action of the operators on the vacuum is given by the following set of equations*

$$\hat{A}(u)|0\rangle = a(u)|0\rangle, \tag{198}$$

$$\hat{D}(u)|0\rangle = d(u)|0\rangle, \tag{199}$$

$$\hat{C}(u)|0\rangle = 0 \tag{200}$$

and the action of $\hat{B}(u)$ on $|0\rangle$ generates the whole Hilbert space $\mathcal{H}$.

Let us write the Lax matrix

$$\hat{L}^i = \hat{L}^{i,+} + \hat{L}^{i,-}, \tag{201}$$

where

$$\hat{L}^{i,+} = \begin{pmatrix} \hat{\alpha}_i & \hat{\beta}_i \\ 0 & \hat{\delta}_i \end{pmatrix} \tag{202}$$

and

$$\hat{L}^{i,-} = \begin{pmatrix} 0 & 0 \\ \hat{\gamma}_i & 0 \end{pmatrix}. \tag{203}$$

Then the monodromy matrix has the form

$$\hat{L}^n\hat{L}^{n-1}\dots\hat{L}^1 = (\hat{L}^{n,+} + \hat{L}^{n,-})(\hat{L}^{n-1,+} + \hat{L}^{n-1,-})\dots(\hat{L}^{1,+} + \hat{L}^{1,-}) = \hat{T}^+ + \hat{Y}. \tag{204}$$

Here

$$\hat{T}^+ = \hat{L}^{n,+}\cdot\hat{L}^{n-1,+}\cdot\dots\cdot\hat{L}^{1,+} \tag{205}$$

and $\hat{Y}$ includes the rest ($2^n - 1$ terms). The operator $\hat{Y}_{ij}$ acts on vacuum by 0

$$\hat{Y}_{ij}|0\rangle = 0,$$

since it contains at least one $\hat{L}^{k,-}$

$$\hat{L}^{k,-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\hat{\gamma}_k. \tag{206}$$

Now we understand that the contribution is given by the positive part of our operators. In case $n = 2$

$$\begin{pmatrix} \hat{\alpha}_2 & \hat{\beta}_2 \\ 0 & \hat{\delta}_2 \end{pmatrix}\begin{pmatrix} \hat{\alpha}_1 & \hat{\beta}_1 \\ 0 & \hat{\delta}_1 \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_2\hat{\alpha}_1 & \hat{\alpha}_2\hat{\beta}_1 + \hat{\beta}_2\hat{\delta}_1 \\ 0 & \hat{\delta}_2\hat{\delta}_1 \end{pmatrix}. \tag{207}$$

The $\hat{\beta}$ operators turn the spin up or down. The multiplication of the matrices could well be continued and we will get the matrix of the asme kind at each step. The action of operators can be now written explicitly

$$a(u) = \prod_{i=1}^{n} \alpha_i(u), \tag{208}$$

$$d(u) = \prod_{i=1}^{n} \delta_i(u). \tag{209}$$

We now see that $\hat{B}$ is a creation operator, whilst $\hat{C}$ is an annihilation operator.

Definition 2. *The substitution*

$$|\Psi\rangle = \hat{B}(u_1)\hat{B}(u_2)\dots\hat{B}(u_M)|0\rangle$$

is called Bethe ansatz. Here $u_1, \dots u_M$ are some variables that are to be determined. The operators $\hat{B}$ correspond to the sector, where M spins are down and $n - M$ spins are up.

Let us now formulate the way how we will find $u_1, \dots, u_M$. Consider

$$(\hat{A}(u) + \hat{D}(u))|\Psi\rangle = (\hat{A}(u) + \hat{D}(u))\hat{B}(u_1)\hat{B}(u_2)\dots\hat{B}(u_M)|0\rangle \tag{210}$$

and commutation relations that we will need later. Now consider

$$R^{\hbar}_{12}(z,w) = 1 + \frac{\hbar}{z-w}P_{12} = 1 + \frac{\hbar}{z-w}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}f(z,w)&0&0&0\\0&1&g(z,w)&0\\0&g(z,w)&1&0\\0&0&0&f(z,w)\end{pmatrix}, \tag{211}$$

where

$$g(z,w) = \frac{\hbar}{z-w} \tag{212}$$

and

$$f(z,w) = \frac{z-w+\hbar}{z-w}. \tag{213}$$

So, we have

$$R^{\hbar}_{12}(z,w)\hat{T}_1(z)\hat{T}_2(w) = \hat{T}_2(w)\hat{T}_1(z)R^{\hbar}_{12}(z,w),$$

where

$$\hat{T}_1(z) = \begin{pmatrix}\hat{A}(z)&0&\hat{B}(z)&0\\0&\hat{A}(z)&0&\hat{B}(z)\\\hat{C}(z)&0&\hat{D}(z)&0\\0&\hat{C}(z)&0&\hat{D}(z)\end{pmatrix} \tag{214}$$

and

$$\hat{T}_2(w) = \begin{pmatrix}\hat{A}(w)&\hat{B}(w)&0&0\\\hat{C}(w)&\hat{D}(w)&0&0\\0&0&\hat{A}(w)&\hat{B}(w)\\0&0&\hat{C}(w)&\hat{D}(w)\end{pmatrix}. \tag{215}$$

One of the most significant relations is

$$[\hat{T}_{ij}(z), \hat{T}_{ij}(w)] = 0 \tag{216}$$

as we get from it that

$$[\hat{A}(u), \hat{A}(u')] = 0, \tag{217}$$

$$[\hat{B}(u), \hat{B}(u')] = 0. \tag{218}$$

There are some non-trivial relations like

$$\hat{A}(v)\hat{B}(u) = f(u,v)\hat{B}(u)\hat{A}(v) + g(v,u)\hat{B}(v)\hat{A}(u). \tag{219}$$

Consider the simple example $M = 1$ (1 spin down):

$$\hat{A}(z)\hat{B}(u_1)|0\rangle = f(u_1, z)\hat{B}(u_1)\hat{A}(z) + g(z, u_1)\hat{B}(z)\hat{A}(u_1)|0\rangle = \tag{220}$$

$$= a(z)f(u_1, z)\hat{B}(u_1)|0\rangle + a(u_1)g(z, u_1)\hat{B}(z)|0\rangle, \tag{221}$$

where $\hat{B}(u_1)|0\rangle = \Psi$ and the second term in the latter equation we call 'unwanted' term - we wish to cancel them out.
In case $M = 2$ $\Psi|0\rangle = \hat{B}(u_1)\hat{B}(u_2)|0\rangle$ and

$$\hat{A}(z)|\Psi\rangle = \hat{A}(z)\hat{B}(u_1)\hat{B}(u_2)|0\rangle = f(u_1, z)\hat{B}(u_1)\hat{A}(z)\hat{B}(u_2)|0\rangle + \tag{222}$$

$$+g(z, u_1)\hat{B}(z)\hat{A}(u_1)\hat{B}(u_2)|0\rangle = f(u_1, z)\hat{B}(u_1)\left(f(u_2, z)\hat{B}(u_2)\hat{A}(z) + g(z, u_2)\hat{B}(z)\hat{A}(u_2)\right) + \tag{223}$$

$$+g(z, u_1)\hat{B}(z)\left(f(u_2, u_1)\hat{B}(u_2)\hat{A}(u_1) + g(u_1, u_2)\hat{B}(u_1)\hat{A}(u_2)\right)|0\rangle = \tag{224}$$

$$= f(u_1, z)f(u_2, z)\hat{B}(u_1)\hat{B}(u_2)\hat{A}(z) + g(z, u_1)f(u_2, u_1)\hat{B}(z)\hat{B}(u_2)\hat{A}(u_1) + \tag{225}$$

$$+\left(f(u_1, z)g(z, u_2) + g(z, u_1)g(u_1, u_2)\right)\hat{B}(z)\hat{B}(u_1)\hat{A}(u_2). \tag{226}$$

Here $f(u_1, z)g(z, u_2) + g(z, u_1)g(u_1, u_2 = g(z, u_2)f(u_1, u_2)$. We see that even in case $M = 2$ the computation is rather complicated. By iterations we get 2^M terms; Firstly, we have

$$|\Psi\rangle = \hat{B}(u_1)\dots\hat{B}(u_M)|0\rangle, \tag{227}$$

then

$$\hat{A}|\Psi\rangle = AB\dots B|0\rangle, \tag{228}$$

$$\hat{A}|\Psi\rangle = BA\dots B|0\rangle, \tag{229}$$

$$\hat{A}|\Psi\rangle = BBA\dots B|0\rangle \tag{230}$$

and so on. We wish to obtain the explicit expression for this procedure. Consider so called off-shell Bethe vector

$$\hat{A}(z)\hat{B}(u_1)\dots\hat{B}(u_M)|0\rangle = a(z)\Lambda(z, u_1, \dots u_M)|\Psi\rangle + \sum_{k=1}^{M} a(u_k)\Lambda(z|\{u\})\hat{B}(z)\hat{B}(z)\prod_{j\neq k}^{M}\hat{B}(u_j)|0\rangle, \tag{231}$$

then we move u_M to the right. So, we have either

$$\hat{B}(u_1)\dots\hat{B}(u_M)\hat{A}(z)|0\rangle \tag{232}$$

or

$$\hat{B}(z)\hat{B}(u_1)\dots\hat{B}(u_M)\hat{A}(u_1)|0\rangle, \tag{233}$$

or

$$\hat{B}(z)\hat{B}(u_1)\hat{B}(u_3)\dots\hat{B}(u_M)\hat{A}(u_2)|0\rangle \tag{234}$$

and all these 3 terms are unwanted. The coefficients have the form

$$\Lambda(z, \{u\}) = \prod_{j=1}^{M} f(u_j, z), \tag{235}$$

$$\Lambda_k(z,\{u\}) = g(z,u_k)\prod_{j\neq k}^{M} f(u_j,u_k), \tag{236}$$

The action of the operator $\hat{D}$ is given by

$$\hat{D}(z)\hat{B}(u_1)\ldots\hat{B}(u_M)|0\rangle = d(z)\widetilde{\Lambda}_k(z,\{u\})|\Psi\rangle + \sum_{k=1}^{M} d(u_k)\widetilde{\Lambda}(u_k)(z,\{u\})\hat{B}(z)\prod_{l\neq k}^{M}\hat{B}(u_l)|0\rangle. \tag{237}$$

We also have the relation

$$\hat{D}(z)\hat{B}(u) = f(z,u)\hat{B}(u)\hat{D}(z) + g(u,z)\hat{B}(z)\hat{D}(u). \tag{238}$$

Hence, we have

$$\widetilde{\Lambda}(z,\{u\}) = \prod_{j=1}^{M} f(u_j,z), \tag{239}$$

$$\widetilde{\Lambda}_k(z,\{u\}) = g(z,u_k)\prod_{j\neq k}^{M} f(u_j,u_k). \tag{240}$$

Eventually, we get Bethe equations, where all unwanted terms vanish

$$(\hat{A}(z)+\hat{D}(z))|\Psi\rangle = a(z)\Lambda(z,\{u\}) + d(z)\widetilde{\Lambda}(z,\{u\})|\Psi\rangle + \tag{241}$$

$$+\sum_{k=1}^{M}\Big(a(u_k)\Lambda_k(z,\{u\}) + d(u_k)\widetilde{\Lambda}_k(z,\{u\}\Big)\times\hat{B}(z)\prod_{k\neq k}^{M}\hat{B}(u_j)|0\rangle. \tag{242}$$

We require for all $k=1,\ldots,M$ (bearing in mind the definitions of Λ and $\widetilde{\Lambda}$

$$a(u_k)\Lambda_k + d(u_k)\widetilde{\Lambda}_k(z|u) = 0. \tag{243}$$

Finally, we have the set of equations

$$a(u_k)\prod_{j\neq k} f(u_j,u_k) = d(u_k)\prod_{j\neq k} f(u_k,u_j) \tag{244}$$

or, equivalently,

$$\frac{a(u_k)}{d(u_k)} = \prod_{j\neq k}\frac{f(u_k,u_j)}{f(u_j,u_k)}. \tag{245}$$

The solutions are called Bethe roots.

Lecture 6

Bethe Ansatz and spin chains

Let us recall the idea discussed in the previous lecture. We had a transfer matrix

$$T(z) = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}. \tag{246}$$

The vector Ψ had the form

$$|\Psi >= \hat{B}(u_1)\dots\hat{B}(u_n)|0> . \tag{247}$$

We used R-matrix elements

$$g(z,w) = \frac{\hbar}{z-w} \to a, \tag{248}$$

$$f(z,w) = \frac{z-w+\hbar}{z-w} \to d. \tag{249}$$

These functions arise in Bethe equation corresponding to M turned spins

$$\frac{a(u_k)}{d(u_k)} = \prod_{j\neq k}\frac{f(u_k,u_j)}{f(u_j,u_k)}. \tag{250}$$

We also introduced the Lax matrix

$$\hat{L}^i(z) = \begin{pmatrix} \hat{\alpha}^i(z) & \hat{\beta}^i(z) \\ \hat{\gamma}^i(z) & \hat{\delta}^i(z) \end{pmatrix}, \tag{251}$$

where $\hat{\alpha}^i|0>=\alpha^i|0>$. The generating function for eigenvalues is given by

$$t(z) = a(z)\Lambda(z|u) + d(z)\widetilde{\Lambda}(z|u). \tag{252}$$

XXX spin chain

Consider the monodromy matrix

$$\hat{T}_0(z) = g_0\hat{L}_0^1(z-z_1)\dots\hat{L}_0^N(z-z_N),$$

where matrix g has the form

$$g = \begin{pmatrix} g_1 & 0 \\ g_0 & g_2 \end{pmatrix} \tag{253}$$

and elements $g_1,\ g_2$ arise in RLL-relations

$$R_n L_1 L_2 = L_2 L_1 R_{12}, \tag{254}$$

$$R_{12} g_1 g_2 = g_2 g_1 R_{12}. \tag{255}$$

Consider the case

$$\hat{L}_0^i(z-z_i) = 1 + \frac{\eta}{z-z_i}\hat{S}_0^i = 1 + \frac{\eta}{z-z_i}P_{0i}, \tag{256}$$

where $\eta \equiv \hbar$. We are looking for

$$\hat{t}(z) = \mathrm{tr}_0\left(g_0\hat{L}_0^1(z-z_1)\dots\hat{L}_0^N(z-z_N)\right). \tag{257}$$

In general form

$$\hat{t}(z) = \mathrm{tr}(g)1 + \sum_{j=1}^{N}\frac{\eta\hat{H}_j}{z-z_j}. \tag{258}$$

We see from this equation that residues corresponding to simple poles are exactly $\eta\hat{H}_j$. The commutation relations are

$$[\hat{H}_j, \hat{H}_k] = 0, \tag{259}$$

which is equivalent to

$$[\hat{t}(z), \hat{t}(w)] = 0. \tag{260}$$

Our goal is to study the spectrum of the Hamiltonian, i.e. solve the equation

$$\hat{H}_m|\Psi >= \lambda_m|\Psi > . \tag{261}$$

Previously the quantum spectral problem was formulated as follows

$$\hat{t}(z)|\Psi >= t(z)|\Psi > . \tag{262}$$

In this case

$$\operatorname*{Res}_{z=z_j} t(z) = \lambda_j\eta$$

(compare with the above one).
We wish to compute

$$\hat{H}_j = \operatorname*{Res}_{z=z_j}\mathrm{tr}_0(g_0R_{01}(z-z_1)\dots R_{0N}(z-z_N)).$$

The residues are known :

$$\operatorname*{Res}_{z=z_i} R_{0i} = \eta P_{0i}.$$

Then

$$\operatorname*{Res}_{z=z_j}\hat{H}_j\,\mathrm{tr}_0(g_0R_{01}(z_j-z_1)\dots R_{0,j-1}(z_j-z_{j-1})P_{0j}R_{0,j+1}(z_j-z_{j+1})\dots R_{0,N}(z_j-z_N)) =$$

$$= \mathrm{tr}_0(P_{0j}g_jR_{j1}(z_j-z_1)\dots R_{j,j-1}(z_j-z_{j-1})\times R_{0,j+1}(z_j-z_{j+1})\dots R_{0N}(z_j-z_N)) = \tag{263}$$

$$= \mathrm{tr}_0(P_{0j}R_{j,j+1}(z_j-z_{j+1})\dots R_{j,N}(z_j-z_N)\times g_jR_{j,1}(z_j-z_1)\dots R_{j,j-1}(z_j-z_{j-1})). \tag{264}$$

Taking into account that

$$\mathrm{tr}(P_{0j}A_j) = A_j, \tag{265}$$

$$\mathrm{tr}_j(P_{0j}A_j) \tag{266}$$

we have

$$\operatorname*{Res}_{z=z_j} \hat{H}_j \operatorname{tr}_0(g_0 R_{01}(z_j - z_1)\dots R_{0,j-1}(z_j - z_{j-1})P_{0j}R_{0,j+1}(z_j - z_{j+1})\dots R_{0,N}(z_j - z_N)) = \tag{267}$$

$$= R_{j,j+1}(z_j - z_{j+1})\dots R_{j,N}(z_j - z_N) \times g_j R_{j,1}(z_j - z_1)\dots R_{j,j-1}(z_j - z_{j-1}). \tag{268}$$

The eigenvalues of $t(z)$ are given by

$$t(z) = a(z)\Lambda(z|u) + d(z)\widetilde{\Lambda}(z|u) \tag{269}$$

and, as we remember,

$$\hat{L}^i(z) = \begin{pmatrix} \hat{\alpha}^i(z) & \hat{\beta}^i(z) \\ \hat{\gamma}^i(z) & \hat{\delta}^i(z) \end{pmatrix}, \tag{270}$$

where $\hat{\alpha}^i|0>=\alpha^i|0>$, $\hat{\delta}^i(z)|0>=\delta^i(z)|0>$. Here

$$\alpha^i(z) = 1 + \frac{\eta}{z - z_i},\ \delta^i(z) = 1. \tag{271}$$

The complete expressions can be written as

$$a(z) = g_1 \prod +i = 1^N \alpha^i(z) = g_1 \prod_{i=1}^{N} \frac{z - z_i + \eta}{z - z_i}, \tag{272}$$

$$d(z) = g_2 \prod_{i=1}^{N} \delta^i(z) = g_2. \tag{273}$$

Then we also have

$$\Lambda(z|u) = \prod_{m=1}^{N} f(u_m, z) = \prod_{m=1}^{M} \frac{u_m - z + \eta}{u_m - z}, \tag{274}$$

$$\widetilde{\Lambda}(z|u) = \prod_{m=1}^{M} \frac{z - u_m + \eta}{z - u_m}. \tag{275}$$

Now we are ready to compute the residues of $t(z)$. The functions $\Lambda, \widetilde{\Lambda}$ are free of poles at z_i, hence, only $a(z)$ contributes:

$$\operatorname*{Res}_{z=z_j} t(z) = g_1 \prod_{k\neq j}^{N} \frac{z_j - z_k + \eta}{z_j - z_k} \prod_{m=1}^{M} \frac{z_j - u_m - \eta}{z_j - u_m}. \tag{276}$$

Now we have to write the Bethe equations:

$$\frac{a(u_m)}{d(u_m)} = \prod_{c\neq m}^{M} \frac{f(u_m, u_c)}{f(u_c, u_m)},\ m = 1,\dots M. \tag{277}$$

On the other hand,

$$\frac{a(u_m)}{d(u_m)} = \frac{g_1}{g_2} \prod_{k=1}^{N} \frac{u_m - z_k + \eta}{u_m - z_k} = \prod_{c\neq m}^{M} \frac{u_m - u_c + \eta}{u_m - u_c - \eta}. \tag{278}$$

Thus, we obtained the complete set of relations for XXX-chain. For sufficiently small η we have so called **Gaudin model**.

Gaudin model ($\eta \to 0$)

Let g be of the form

$$g = e^{\eta v} = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \tag{279}$$

where v is a matrix of the same form as g. Then $g = 1 + \eta v$. In this case

$$\hat{H}_j = (1 + \eta r_{j,j+1}) \dots (1 + \eta r_{j,N})(1 + \eta v)(1 + \eta r_{j,1}) \dots (1 + \eta r_{j,j-1}). \tag{280}$$

We want to compute the linear (with respect to η) non-trivial contribution:

$$\hat{H}_j = 1 + \eta(H^a) + O(\eta^2), \tag{281}$$

where H^a is a Gaudin Hamiltonian of the form

$$H_j^a = v + \sum_{k \neq j}^{N} r_{jk}(z_j - z_k) = v + \sum_{k \neq j}^{N} \frac{P_{jk}}{z_j - z_k}. \tag{282}$$

The Gaudin Hamiltonians commute

$$[H_j^a, H_k^a] = 0, \tag{283}$$

which follows from the classical r-matrix structure. The same could be obtained from considering limit $\eta \to 0$. Eigenvalues for Gaudin Hamiltonian can be written as

$$h_1^G = v + \sum_{k \neq i}^{n} \frac{\hbar}{z_j - z_k} + \sum_{\gamma=1}^{M} \frac{\hbar}{u_\gamma - z_i}, \tag{284}$$

$$2v + \sum_{k=1}^{N} \frac{\hbar}{u_\gamma - z_k} = 2\hbar \sum_{\alpha \neq \gamma}^{M} \frac{\hbar}{u_\alpha - u_\gamma}. \tag{285}$$

Lecture 7

KZ equations and Gaudin model

The quantum Gaudin model is defined by

$$\hat{H}_i^G = v^i + \sum_{j\neq i} r_{ij}(z_i - z_j),$$

where $r_{ij}(z) = \frac{P_{ij}}{z}$. Let us now describe the Gaudin model at classical level. Consider the matrix-valued function

$$L(z) = \sum_{a=1}^{n} \frac{S^a}{z - z_a} + v,$$

where S^a is a matrix with elements $\{S^a_{ij}\}_{i,j=1}^N$. The Hamiltonians are generated by traces (Casimir functions in quadratic term)

$$H(z) = \frac{1}{2}\operatorname{tr}(L^2(z)) = \frac{1}{2}\sum_{a=1}^{n} \frac{\operatorname{tr}((S^a)^2)}{(z-z_a)^2} + \sum_{a=1}^{n} \frac{H_a}{z - z_a} + \operatorname{tr}(v^2), \tag{286}$$

where H^a is a classical Gaudin Hamiltonian and

$$\{S^a_{ij}, S^b_{kl}\} = \delta^{ab}(S^a_{il}\delta_{kj} - S^a_{kj}\delta_{il}). \tag{287}$$

Hence, we deal with the points on Riemann sphere and assign to each point a degree of freedom S^1, S^2 and so on. Let us write the expression for the classical Hamiltonian

$$H_a = \sum_{a\neq b}^{n} \frac{\operatorname{tr}(S^a S^b)}{z_a - z_b} + \operatorname{tr}(S^a v). \tag{288}$$

The equations of motion then have the form

$$\begin{cases} \frac{dS^a}{dt_a} = \frac{[S^b, S^a]}{z_b - z_a}, \ a \neq b, \\ \frac{dS^a}{dt_a} = -\sum\limits_{c\neq a} \frac{[S_c, S_a]}{z_c - z_a}. \end{cases} \tag{289}$$

In terms of the Lax representation

$$\frac{dL(z)}{dt_a} = [L(z), M_a(z)], \tag{290}$$

where $M_a(z) = -\frac{S^a}{z - z_a}$.

Consider now the Hilbert space of states of the form $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N \cdots \otimes \mathbb{C}^N$. The quantisation procedure implies

$$S^i_{ab} \to \hat{S}^i_{ab} = 1 \otimes \hbar E_{ba} \otimes \cdots \otimes 1.$$

Then

$$\hat{H}^i = \sum_{i\neq j}^{n} \frac{\mathrm{tr}(S^i S^j)}{z_i - z_j} + \mathrm{tr}(S^i v) = \sum_{i\neq j}^{n} \sum_{a,b=1}^{N} \frac{S^i_{ab} S^j_{ba}}{z_i - z_j} + \sum_{a,b=1}^{N} S^i_{aa} v_a,$$

where $v = \mathrm{diag}(v_1, \dots v_N)$. We can rewrite the expression as

$$\hat{H}_i = \sum_{j\neq i}^{n} \sum_{a,b=1}^{N} \frac{E^i_{ba} E^j_{ab}}{z_i - z_j} + \sum_{a=1}^{N} E^i_{aa} v^a \tag{291}$$

and we know that

$$\frac{E^i_{ba} E^j_{ab}}{z_i - z_j} = \frac{P_{ij}}{z_i - z_j}. \tag{292}$$

The Planck's constant can be either put or ignored, it does not matter much. Remember that we discussed Schroedinger equation for Gaudin model

$$\hat{H}^i_G |\Psi >= E_i |\Psi > . \tag{293}$$

Now we are going to consider Knizhnik-Zamolodchikov (KZ) equations

$$\partial_{z_i} |\Psi >= \hat{H}^G_i |\Psi > . \tag{294}$$

Here we can use the covariant KZ derivative $\nabla_i := \partial_{z_i} - \hat{H}^G_i$, which satisfies

$$[\hat{\nabla}_i \hat{\nabla}_j] = [\partial_{z_i} - \hat{H}^G_i, \partial_{z_j} - \hat{H}^G_j] = \partial_{z_j} \hat{H}^G_i - \partial_{z_i} \hat{H}^G_j. \tag{295}$$

In detail,

$$\hat{H}^G_i = r_{i1} + r_{i2} + \dots + r_{i,i-1} + r_{i,i+1} + \dots + r_{i,n},$$
$$\hat{H}^G_j = r_{j1} + r_{j2} + \dots + r_{j,j-1} + r_{j,j+1} + \dots + r_{j,n}.$$

Therefore, commutator can be rewritten as

$$[\hat{\nabla}_i \hat{\nabla}_j] = \partial_{z_j} r_{ij} - \partial_{z_i} r_{ji} = (\partial_{z_i} + \partial_{z_j}) r_{ij} = 0. \tag{296}$$

By replacing $t \to z$ at (289) we obtain *non-autonomous* system or Schlesinger system. Consider now quantum KZ equations. When discussing spin chains, we introduced

$$\hat{t}(z) = 1 \cdot \mathrm{tr}(g) + \eta \sum_{j=1}^{n} \frac{\hat{H}_j(z)}{z - z_j}, \tag{297}$$

where

$$\hat{H}_i = R_{i,i-1} \dots R_{i,1} g^i R_{i,n} \dots R_{i,i+1}. \tag{298}$$

The shift operators are defined as

$$T_i f(z_1, \dots z_n) = f(z_1, \dots z_i + \eta\hbar, z_{i+1}, \dots, z_n). \tag{299}$$

We wish to discuss the compatible set of equations

$$T_i |\Psi >= K^\hbar_i |\Psi >, \tag{300}$$

where

$$K_i^\hbar = R^\hbar_{i,i-1}(z_i-z_{i-1}+\eta\hbar)\dots R^\hbar_{i1}(z_i-z_1+\eta\hbar)g^i\times R^{-1}_{ni}(z_n-z_i)\dots R^{-1}_{i+1,i}(z_{i+1}-z_i). \quad (301)$$

Suppose our R-matrix is unitary, then

$$R_{ij}R_{ji}=1. \quad (302)$$

Compatibility of equations implies that

$$T_jT_i|\Psi>=T_jK_i^\hbar|\Psi>=(T_jK_i^\hbar)T_j|\Psi>=(T_jK_i^\hbar)K_j^\hbar|\Psi>. \quad (303)$$

Hence, the compatibility condition can be written as

$$(T_jK_i^\hbar)K_j^\hbar=(T_iK_j^\hbar)K_i^\hbar. \quad (304)$$

Let us now consider the example $n=3$:

$$K_1^\hbar = g^1R_{31}^{-1}R_{21}^{-1}, \quad (305)$$

$$K_2^\hbar = R_{21}g^2R_{32}^{-1}, \quad (306)$$

$$K_3^\hbar = R_{32}R_{31}g^3. \quad (307)$$

The compatibility condition then is of the form

$$(T_2K_1^\hbar K_2^\hbar=(T_1K_2^\hbar)K_1^\hbar. \quad (308)$$

The left-hand side takes the form

$$g^1R_{31}^{-1}(R_{21})^{-1}R_{21}g^2R_{32}^{-1}=g^1g^2R_{31}^{-1}R_{32}^{-1}, \quad (309)$$

whilst the right-hand side

$$R_{21}g^2R_{32}^{-1}g^1R_{31}^{-1}R_{21}^{-1}=R_{21}g^2g^1R_{32}^{-1}R_{31}^{-1}R_{21}^{-1}=g^1g^2R_{21}R_{32}^{-1}R_{31}^{-1}R_{21}^{-1}. \quad (310)$$

By multiplying both expressions by R_{21}^{-1}, we have

$$R_{32}R_{31}R_{21}=R_{21}R_{31}R_{32}, \quad (311)$$

from where, by changing $3\to1$, we obtain the Yang-Baxter equation.

Quantum Calogero-Moser model

Consider the Hamiltonian

$$\hat{H}^{CM}=\hbar^2\sum_{i=1}^n\partial_{x_i}^2-\sum_{i\neq j}\frac{\nu(\nu-\hbar)}{(x_i-x_j)^2}. \quad (312)$$

In classical case we had

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{\nu^2}{2}\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}. \tag{313}$$

KZ equations can be written as

$$\hbar\partial_{x_i}|\varphi >= \left(\nu\sum_{i\neq j}\frac{p_{ij}}{x_i - x_j} + g^i\right)|\varphi >= \hat{M}_a^i|\varphi > . \tag{314}$$

Here $g = \mathrm{diag}(g_1, \dots g_n)$,

$$\hat{M}_a = \sum_{i=1}^{n} E_{aa}^{(i)}, \tag{315}$$

such operator measures the number of spins being "up" or "down". The index i denotes the component. For instance, when $N = 2$,

$$\hat{M}_1 = E_{11}^{(1)} + \dots E_{11}^{(n)}, \tag{316}$$

$$\hat{M}_2 = E_{22}^{(1)} + \dots E_{22}^{(n)}, \tag{317}$$

Note that

$$[\hat{H}_i^a, \hat{M}_a] = 0. \tag{318}$$

Consider the eigenvalue problems for these commuting operators

$$\hat{H}_i|\varphi >= H_i|\varphi >, \tag{319}$$

$$\hat{M}_a|\varphi >= M_a|\varphi > . \tag{320}$$

The Hilbert space of states can be described as

$$\mathcal{H} = (\mathbb{C}^N)^{\otimes n} = \bigoplus_{M_1,\dots M_N} \nu(\{M_i\}). \tag{321}$$

Consider

$$\begin{aligned}\hbar^2\partial_{x_i}^2|\varphi >= -\hbar\nu\sum_{j\neq i}\frac{P_{ij}|\varphi >}{(x_i - x_j)^2} + \left(\nu\sum_{i\neq j}\frac{P_{ij}}{(x_i - x_j)} + g^{(i)}\right)\left(\nu\sum_{i\neq l}\frac{P_{il}}{(x_i - x_l)} + g^{(i)}\right)|\varphi >= \\ = -\hbar\nu\sum_{i\neq j}\frac{P_{ij}}{(x_i - x_j)^2}|\varphi > + (g^{(i)})^2 + \left(g^{(i)}\sum_{l\neq i}\frac{\nu P_{ij}}{x_i - x_l} + \sum_{j\neq i}\frac{P_{ij}}{x_i - x_j}\right) + \\ + \left(\sum_{j\neq i}\frac{\nu^2 P_{ij}^2}{x_i - x_j} + \sum_{l\neq j, l, j\neq i}\frac{P_{ij}P_{il}}{(x_i - x_j)(x_i - x_l)}\right)|\varphi > .\end{aligned} \tag{322}$$

Now we define Matsuno-Cherednik projection

$$< \Omega|\varphi >, \tag{323}$$

where

$$\Omega = \sum_J < J|. \tag{324}$$

Here $< J|$ are basis vectors in $\mathcal{H}^*$. The action on a permutation operator is trivial

$$< \Omega|P_{ij} =< \Omega|. \tag{325}$$

With the help of bra- vector Ω all permutation operators can be removed from the above expressions. Hence, we get

$$< \Omega| \sum \hbar^2 \partial^2_{x_i} |\varphi >= \sum_{i \neq j} \frac{\nu^2 - \nu\hbar}{(x_i - x_j)^2} + \sum_{i=1}^{n} < \Omega| g^{(ii)}|^2 \varphi >, \tag{326}$$

where

$$g^{ii} = \sum_{i=1}^{L} \sum_{a=1}^{N} < \Omega| E^{(i)}_{aa} g^2_a |\varphi >= \sum_{a=1}^{N} < \Omega| \hat{M}_a g^2_a |\varphi >= \Psi \sum_{a=1}^{N} M_a g^2_a. \tag{327}$$

Therefore, we obtained the energy

$$E = \sum_{a=1}^{N} M_a g^2_a. \tag{328}$$

Lecture 8

Quantum Calogero-Moser model

The Hamiltonian for the quantum model has the form

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{i=1}^{N}\partial_{q_i}^2 + g\sum_{i\neq j}\frac{1}{(q_i-q_j)^2} + \frac{\omega}{2}\sum_{i=1}^{N}q_i^2, \tag{329}$$

where $g=\nu(\nu-\hbar)$. In case $N=1$ we have

$$\hat{H} = -\frac{\hbar^2}{2}\partial_q^2 + \frac{m\omega^2}{2}, \tag{330}$$

which can be expressed with the help of annihialation and creation operators correspondingly

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hbar\partial_q + \omega q\right), \tag{331}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\hbar\partial_q + \omega q\right). \tag{332}$$

Now we can rewrite the Hamiltonian as

$$\hat{H} = \hat{a}^\dagger\hat{a} + E_0, \tag{333}$$

where $E_0 = \frac{\hbar\omega}{2}$ is the energy of ground state and the corresponding ground state function is

$$\Delta(q) = \exp\left(-\frac{\omega q^2}{2\hbar}\right). \tag{334}$$

with the following property $\hat{a}\Delta(q)=0$. Define

$$\hat{H}' = \Delta^{-1}(\hat{H}-E_0)\Delta = \Delta^{-1}\hat{a}^\dagger\hat{a}\Delta = \frac{\hbar^2\partial_q^2}{2} + \hbar\omega q\partial_q. \tag{335}$$

Our goal is to find the eigenvalues (spectrum) of $\hat{H}'$ in the space of symmetric polynomials generated by q $\{1, q, q^2, \dots q^n\}$. For this basis we use the standard notation $e_1, e_2, \dots$. Consider

$$\Phi_n = \sum_{k=0}^{n} c_k q^{n-k} = c_0 q^n + c_1 q^{n-1} + \dots \tag{336}$$

Here $c_0 \neq 0$.The spectral problem has the form

$$\hat{H}'\Phi_n = -\frac{\hbar^2}{2}\sum_{k=0}^{n-2} c_k(n-k)(n-k-1)q^{n-k-2} + \hbar\omega\sum_{k=1}^{n} c_k(n-k)q^{n-k} = E_n'\Phi_n. \tag{337}$$

Let us reduce the task to $n=2$:

$$\Phi_2 = c_0q^2 + c_1q + c_2, \tag{338}$$

then

$$\hat{H}'\Phi_2 = -\hbar^2 c_0 + 2c_0\hbar\omega q^2 + c_1\hbar\omega q = E_2'(c_0 q^2 + c_1 q + c_2). \tag{339}$$

Hence, we obtain the following equations (comparing coefficients corresponding to powers):

$$q^2 : c_0 E_2' = 2c_0\hbar\omega,\ \ E_2' = 2\hbar\omega, \tag{340}$$

$$q : c_1 E_2' = \hbar\omega c_1, \tag{341}$$

so $c_1 = 0$, otherwise we arrive to contradiction. The last equation takes the form

$$-\hbar^2 c_0 = c_2 E_2, \tag{342}$$

hence, $c_2 = \frac{\hbar c_0}{2\omega}$.
Let us return to the general case. We can see from the previous calculation that $c_k = 0$, if k is odd. For all even k we have a recursive formula

$$c_k = -\frac{\hbar}{2\omega}\frac{(n-k+1)(n-k+2)}{k}c_{k-2}. \tag{343}$$

Choose c_0 such that

$$c_0 = 2^n\left(\frac{\omega}{\pi}\right)^{n/2}. \tag{344}$$

Therefore,

$$c_k = \left(-\frac{\hbar}{\omega}\right)^{k/2}\frac{n!}{(n-k)!}\frac{c_0}{2^k\Gamma(\frac{k}{2}+1)}. \tag{345}$$

Finally, we obtain the eigenfunctions, which are exactly the Hermitean polynomials

$$\Phi_n = H_n\left(\sqrt{\frac{\omega}{\hbar}}\right)q. \tag{346}$$

But these eigenfunctions correspond to $\hat{H}'$. For the original Hamiltonian we obtain

$$\Psi_n = \Delta\Phi_n, \tag{347}$$

$$E_n = E_n' + E_0 = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{348}$$

Proposition 4. *The Hamiltonian can be represented as*

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}\hat{A}_i^\dagger\hat{A}_i + E_0, \tag{349}$$

where

$$E_0 = \frac{1}{2}N\omega(\hbar + (N-1)\nu), \tag{350}$$

$$\hat{A}_i = \hbar\frac{\partial}{\partial q_i} + \omega q_i - \sum_{j\neq i}^{n}\frac{\nu}{q_i - q_j}, \tag{351}$$

$$\hat{A}_i^\dagger = -\hbar\frac{\partial}{\partial q_i} + \omega q_i - \sum_{j\neq i}^{n}\frac{\nu}{q_i - q_j}. \tag{352}$$

For example, when $N = 2$,

$$\hat{A}_1^\dagger \hat{A}_1 = \left(-\hbar\partial_1 + \omega q_1 - \frac{\nu}{q_1 - q_2}\right)\left(\hbar\partial_1 + \omega q_1 - \frac{\nu}{q_1 - q_2}\right) = \tag{353}$$

$$= -\hbar^2\partial_1^2 - \hbar\omega - \frac{\hbar\nu}{(q_1 - q_2)^2} - \left(\omega q_1 - \frac{\nu}{q_1 - q_2}\right)\hbar\partial_1 + \left(\omega q_1 - \frac{\nu}{q_1 - q_2}\right)\hbar\partial_1 +$$

$$+ \left(\omega q_1 - \frac{\nu}{q_1 - q_2}\right)^2 = -\hbar^2\partial_1^2 + \omega^2 q_1^2 + \frac{\nu^2 - \nu\hbar}{(q_1 - q_2)^2} - \frac{2\nu\omega q_1}{q_1 - q_2} - \hbar\omega.$$

Consider now

$$\hat{A}_2^\dagger \hat{A}_2 = -\hbar^2\partial_2^2 + \omega^2 q_2^2 + \frac{\nu^2 - \nu\hbar}{(q_2 - q_1)^2} - \frac{2\nu\omega q_2}{q_2 - q_1} - \hbar\omega.$$

By summing these results up, we obtain our Hamiltonian. The ground state is given by the Laughlin function

$$\Delta(q) - \prod_{i=1}^{N} e^{-\frac{\omega q_i^2}{2\hbar}} \cdot \prod_{i<j}^{N} (q_i - q_j)^\beta, \tag{354}$$

where $\beta = \frac{\nu}{\hbar}$. From the definition it is easy to see

$$\hat{A}_i \Delta(q) = 0. \tag{355}$$

After applying the conjugation to the Hamiltonian, we get

$$\hat{H}' = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q_i^2} + \hbar\omega q_i \frac{\partial}{\partial q_i}\right) - \hbar\nu \sum_{i<j} \frac{1}{q_i - q_j}\left(\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_j}\right). \tag{356}$$

Now let us get back to the eigenvalue problem

$$\hat{H}'\Phi_n = E_n \Phi_n \tag{357}$$

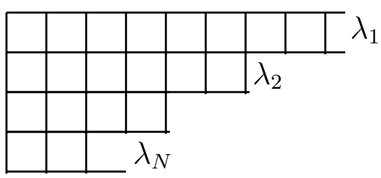
for the partition q^λ given by

$$q^\lambda = q_1^\lambda \ldots q_n^{\lambda_N}, \ \lambda_i \in \mathbb{Z}_{\geqslant 0} \tag{358}$$

bearing in mind that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ We also define

$$|\lambda| = \sum_{i=1}^{k} \lambda_i, \tag{359}$$

where k is a length of partition. We relate the Young diagram to each partition correspondingly



So, we search for solutions in the basis formed by the symmetric polynomials. Conisder, for simplicity, the following monomial (being symmetrised to satisfy the conditions of our problem)

$$q_i^{\lambda_i} q_j^{\lambda_j} + q_i^{\lambda_j} q_j^{\lambda_i}. \tag{360}$$

We have to compute

$$\frac{1}{q_i - q_j}\left(\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_j}\right)(q_i^{\lambda_i} q_j^{\lambda_j} + q_i^{\lambda_j} q_j^{\lambda_i}) = \frac{1}{q_i - q_j}\left(\lambda_i q_i^{\lambda_i - 1} q_j^{\lambda_j} - \lambda_j q_i^{\lambda_i} q_j^{\lambda_j - 1} + \lambda_j q_i^{\lambda_j - 1} q_j^{\lambda^j} - \lambda_i q_i^{\lambda_j} q_j^{\lambda_i - 1}\right) = \tag{361}$$

$$= \frac{1}{q_i - q_j}\left(\lambda_i q_i^{\lambda_j} q_j^{\lambda_j}\left(q_i^{\lambda_i - \lambda_j - 1} - q_j^{\lambda_i - \lambda_j - 1}\right) - \lambda_j q_i^{\lambda_j - 1} q_j^{\lambda_j - 1}\left(q_i^{\lambda_i - \lambda_j - 1} - q_j^{\lambda_i - \lambda_j - 1}\right)\right).$$

Now, by using the obvious identity

$$\frac{x^n - y^n}{x - y} = x^{n-1} + y x^{n-2} + \dots + y^{n-1} \tag{362}$$

we get

$$\frac{1}{q_i - q_j}\left(\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_j}\right)(q_i^{\lambda_i} q_j^{\lambda_j} + q_i^{\lambda_j} q_j^{\lambda_i}) = \lambda_i q_i^{\lambda_j} q_j^{\lambda_j} \sum_{k=1}^{\lambda_i - \lambda_j - 1} q_i^{\lambda_i - \lambda_j - 1 - k} q_j^{k-1} - \lambda_j q_i^{\lambda_j - 1} q_j^{\lambda_j - 1} \sum_{k=0}^{\lambda_i - \lambda_j} q_i^{\lambda_i - \lambda_j - k} q_j^{k}. \tag{363}$$

By simplifying the expression, we get

$$\frac{1}{q_i - q_j}\left(\frac{\partial}{\partial q_i} - \frac{\partial}{\partial q_j}\right)(q_i^{\lambda_i} q_j^{\lambda_j} + q_i^{\lambda_j} q_j^{\lambda_i}) = \lambda_i \sum_{k=1}^{\lambda_i - \lambda_j - 1} q_i^{\lambda_i - 1 - k} q_j^{\lambda_j + k - 1} - \lambda_j \sum_{k=0}^{\lambda_i - \lambda_j} q_i^{\lambda_i - k - 1} q_j^{\lambda_j + k - 1} = \tag{364}$$

$$= (\lambda_i - \lambda_j) \sum_{k=1}^{\lambda_i - \lambda_j - 1} q_i^{\lambda_i - 1 - k} q_j^{\lambda_j + k - 1} - \lambda_j (q_i^{\lambda_i - 1} q_j^{\lambda_j - 1} + q_i^{\lambda_j - 1} q_j^{\lambda_i - 1}).$$

As the basis in the space of symmetric polynomials, we may consider

1) $p_i = \sum\limits_{k=1}^{N} q_k^i$,
2) elementary symmetric functions $e_k = \sum\limits_{i_1 < i_2 < \dots i_k} q_{i_1} \dots q_{i_k}$.

For the (1) we have a determinant formula

$$p_k = \det \begin{pmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \dots & 0 \\ 3e_3 & e_2 & e_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ ke_k & e_{k-1} & e_{k-2} & \dots & e_1 \end{pmatrix}. \tag{365}$$

In case $N = 2$ we can easily see that $p_1 = q_1 + q_2$, $p_2 = q_1^2 + q_2^2$ and $p_2 = e_1^2 - 2e_2$. We use the basis of monomial symmetric functions

$$m_\lambda \sum_{\sigma \in S_N} q_{\sigma(1)}^{\lambda_1} q_{\sigma(2)}^{\lambda_2} \dots q_{\sigma(N)}^{\lambda_N}. \tag{366}$$

Theorem 3. *Thea action of the Hamiltonian is given by*

$$\hat{H}' m_\lambda = \hbar\omega|\lambda| m_\lambda + \sum_{\mu<\lambda} C_{\lambda\mu} m_\mu, \tag{367}$$

where $\mu < \lambda$ *is a dominant partial order and* $\mu_1 + \ldots \mu_j < \lambda_1 + \ldots \lambda_j$, $\forall j$. *The action of* $\hat{H}'$ *on the* m_λ *is given by the triangular matrix (low triangular, to be precise). Consequently,*

$$E'_\lambda = \hbar\omega|\lambda|. \tag{368}$$

In case of the original Hamiltonian,

$$E_\lambda = E'_\lambda + E_0 = \hbar\omega \sum_{j=1}^{N} \lambda_j + \frac{N\hbar\omega}{2} + \frac{N(N-1)\nu\omega}{2}. \tag{369}$$

Eigenvectors can be found by considering the generalised Hermitean polynomials

$$m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu. \tag{370}$$

Lecture 9

Generalisations of the Quantum CM Model

In the classical formulation the Hamiltonian of the Calogero-Moser Model has the form

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{\nu^2}{2} \sum_{i \neq j} U(q_i - q_j), \tag{371}$$

where the potential corresponds to different models:

$$U(q) = \begin{cases} \frac{\nu}{q^2}, \text{ Rational CM model,} \\ \frac{\nu}{\sin^2 q}, \text{ Calogero} - \text{Sutherland,} \\ \nu \mathcal{P}(q), \text{ elliptic CM model.} \end{cases} \tag{372}$$

Here $\mathcal{P}(q)$ is the elliptic Weierstrass function. Im quantum case the CM model is defined by

$$\hat{H} = \frac{-\hbar^2}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{4l^2 \sin^2 \frac{q_i - q_j}{2l}}, \tag{373}$$

where $g = \nu(\nu - \hbar)$ and $Q_j = e^{\frac{iq_j}{l}}$. We can also use the Euler's formula

$$\sin \frac{q_i - q_j}{2l} = \frac{1}{2i} \left(e^{i\frac{q_i - q_j}{2l}} - e^{-\frac{q_i - q_j}{2l}} \right) = \frac{1}{2i} \left(\sqrt{\frac{Q_i}{Q_j}} - \sqrt{\frac{Q_j}{Q_i}} \right) = \frac{1}{2i} \cdot \frac{Q_i - Q_j}{\sqrt{Q_i Q_j}}. \tag{374}$$

Then it is easy to see that

$$\sin^2 \frac{q_i - q_j}{2l} = -\frac{1}{4} \frac{Q_i - Q_j}{Q_i Q_j}. \tag{375}$$

Now let us rewrite the Hamiltonian in the following way

$$\hat{H} = \sum_{j=1}^{N} A_i A_i^\dagger + E_0, \tag{376}$$

where the ground state energy is given by

$$E_0 = \frac{\nu^2}{l^2} \frac{N(N^2 - 1)}{24}. \tag{377}$$

The ground state is similar to one defined in the previous lecture

$$\Delta = \prod_{i<j}^{N} (Q_i - Q_j)^\beta \cdot \prod_{i=1}^{N} Q_i^{-\frac{1}{2}(N-1)\beta} \tag{378}$$

and the annihilation condition is satisfied

$$A_i \Delta = 0. \tag{379}$$

The explicit formulas for operators are

$$A_i = \hbar\frac{\partial}{\partial q_i} - \frac{\nu}{2l}\sum_{k:i\neq k}^{N} \cot\frac{q_i - q_j}{2l}, \tag{380}$$

$$A_i^\dagger = -\hbar\frac{\partial}{\partial q_i} - \frac{\nu}{2l}\sum_{k:i\neq k}^{N} \cot\frac{q_i - q_j}{2l}. \tag{381}$$

Now let us conjugate the Hamiltonian with the ground state

$$\hat{H} \to \hat{H}' = \frac{2l^2}{\hbar^2}\Delta^{-1}(\hat{H} - E_0)\Delta = \sum_{i=1}^{N}\left(Q_i\frac{\partial}{\partial Q_i}\right)^2 + \beta\sum_{i<j}\frac{Q_i+Q_j}{Q_i-Q_j}\left(Q_i\frac{\partial}{\partial Q_i} - Q_j\frac{\partial}{\partial Q_j}\right). \tag{382}$$

Hence, we deal with the second-order differential operator. Consider monomial symmetric polynomials for the partition $\lambda = (\lambda_1, \lambda_2 \dots, \lambda_N)$

$$m_\lambda = \sum_{\sigma\in S_N} Q_{\sigma(1)}^{\lambda_1} Q_{\sigma(2)}^{\lambda_2} \dots Q_{\sigma(N)}^{\lambda_N}. \tag{383}$$

Define Jack polynomials now

$$J_\lambda(Q,\beta) = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu}^{(\beta)} m_\mu. \tag{384}$$

Theorem 4. *For the Hamiltonian (382) the exists a set of symmetric functions J_λ, which satisfy the equation*

$$\hat{H}' J_\lambda = \varepsilon_\lambda J_\lambda \tag{385}$$

for

$$\varepsilon_\lambda = \sum_i \lambda_i^2 + \beta\sum_{i<j}(\lambda_i - \lambda_j). \tag{386}$$

Finally, we have

$$\hat{H}\Psi_\lambda(q) = E_\lambda\Psi_\lambda(q), \tag{387}$$

where

$$\Psi_\lambda(q) = \left(\prod_{i=1}^{N} Q_i\right)^{-(N-1)\frac{\beta}{2}} \prod_{i<j}(Q_i - Q_j)^\beta J_\lambda(Q,\beta) \tag{388}$$

and

$$E_\lambda = \frac{\nu^2}{l^2}\cdot\frac{N(N^2-1)}{24} + \frac{\hbar^2}{2l^2}\sum_{j=1}^{N}(\lambda_j + \beta(N+1-2j)))\lambda_j. \tag{389}$$

In other words,

$$E_\lambda = \frac{1}{2}\sum_i p_i^2, \tag{390}$$

where p_i is a quasi momentum

$$p_i = \frac{\hbar}{l}(\lambda_{\mu+1-i} + (i - \frac{N+1}{2})\beta). \tag{391}$$

So far, we have studied the Lax representation, but most of the time we have been dealing with something else. Now, in the quantum case, we have

$$\hat{L}_{ij} = -\hbar\delta_{ij}\frac{\partial}{\partial q} + (1-\delta_{ij})\frac{\nu}{q_i - q_j}. \tag{392}$$

The quantum Lax equation has the form

$$[\hat{H}, \hat{L}] = [\hat{L}, \hat{M}] \tag{393}$$

or, equivalently,

$$[\hat{H}, \hat{L}_{ij}] = [\hat{L}, \hat{M}]_{ij} = \sum_{k=1}^{N} \hat{L}_{ik}\hat{M}_{kj} - \hat{M}_{ik}\hat{L}_{kj}, \tag{394}$$

where $M_{ij} = d_i\delta_{ij} - (1-\delta_{ij})\frac{\nu}{(q_i-q_j)^2}$. The problem is that we cannot compute the trace, though we can write

$$[\hat{H}, \hat{L}^k] = [\hat{L}^k, \hat{M}]. \tag{395}$$

In the quantum case we have

$$\mathrm{tr}(AB) = \sum A_{ij}B_{ji} \neq \sum B_{ji}A_{ij} = \mathrm{tr}(AB). \tag{396}$$

Yet we have

$$\sum_{j=1}^{N} M_{ij} = \sum_{i=1}^{N} M_i j = 0. \tag{397}$$

The total sum (ts)

$$tsL^k = \sum_{i,j=1}^{N} (L^k)_{ij} \tag{398}$$

can be used for the representation (395) in order to compute the conserved quantities. In the classical case we want to compute $\mathrm{tr}(L^k)$ and we want to consider a characteristic polynomial

$$\det(1w + L) = 0. \tag{399}$$

In the quantum case we need to introduce the time ordering operation (: :), where the momenta go to the left, whilst the coordinates go to the right

$$: \det(1\lambda + \hat{L}) := \sum_i \lambda^i \hat{\hat{H}}^i. \tag{400}$$

Theorem 5.

$$\Delta^{-1} : \det(w + \hat{L}) : \Delta = \hat{D}(w) = \frac{1}{\prod_{i<j}(Q_i - Q_j)}. \tag{401}$$

$$\cdot \sum_{\sigma \in S_N} (-1)^{|\sigma|} \prod_{i=1}^{N} \left\{ Q_i^{N-\sigma(i)} \left(w + \frac{\nu}{l} (\sigma(i) - \frac{N+1}{2} + i\hbar \frac{\partial}{\partial q_i} \right) \right\}.$$

Moreover,

$$\hat{D}(w) J_\lambda(Q, \beta) = \Lambda(w) J_\lambda(Q, \beta), \tag{402}$$

where $\Lambda(w) = \prod_{j=1}^{N} (w - p_j)$.

In some sense, we quantised the characteristic polynomial.

Trigonometric Ruisenaars-Schneider model

Consider the Hamiltonian

$$\hat{H}_k = \frac{1}{2} \left(S\hat{S}_k + \hat{S}_{-k} \right), \tag{403}$$

where difference operators $\hat{S}_k$ are defined as

$$\hat{S}_{\pm k} = \sum_{J \in \{1,\dots,N\} |J|=k} \prod_{i \in J,\ j \notin J} h^{\pm}(q_i - q_j) \left(\prod_{s \in J} T_s^{\pm} \right) \prod_{i \in J,\ j \notin J} h^{\pm}(q_j - q_i), \tag{404}$$

where

$$h^{\pm} = \left(\frac{\sin \frac{1}{2l} (q_i - q_j \pm i\nu)}{\sin \frac{1}{2l} (q_i - q_j)} \right)^{1/2} \tag{405}$$

and shift operators are given by

$$T_k^{\pm} = e^{\frac{i}{l} \frac{\partial}{\partial q_k}} \tag{406}$$

and their action

$$T_k^{\pm} f(Q_1, Q_2, \dots Q_N) = f(Q_1, \dots Q_{k-1} Q_k q^{k-1}, Q_{k+1}, \dots Q_N). \tag{407}$$

This is a generalisation of the CM model. In case $k = 1$,

$$\hat{S}_1 = \sum_{j=1}^{N} \prod_{i: i \neq j} h^+(q_i - q_j) \hat{T}_j \prod_{i: i \neq j} h^+(q_j - q_i). \tag{408}$$

In the classical case $T_j = e^{p_j/c}$ and

$$S_1 = \sum_{j=1}^{N} \prod_{i: i \neq j} h^+(q_i - q_j) h^+(q_j - q_i) e^{p_j/c} \tag{409}$$

and

$$h^+(q_i - q_j) h^+(q_j - q_i) = \sqrt{1 - \frac{\sin^2 (i/2l\nu)}{\sin^2 (q_i - q_j)/2l}}. \tag{410}$$

By inserting ε into the exponent and the sin function in the enumerator and considering the limit $\varepsilon \to 0$ we get

$$S_1 = N + \frac{\varepsilon}{c} \sum_i p_i + \varepsilon^2 \left(\frac{1}{l^2} \sum_i p_i^2 + \sum_{i \neq j} \frac{\nu^2}{\sin^2 \frac{q_i - q_j}{2l}} \right). \tag{411}$$

Lecture 10

Ruseinaars-Schneider model

Today we are going to discuss the trigonometric Ruseinaars-Schneider model defined by the Hamiltonian

$$\hat{H}_k = \frac{1}{2}\left(\hat{S}_k + \hat{S}_{-k}\right),$$

where

$$\hat{S}_{\pm k} = \sum_{|J|=k} \prod_{i\in J,\ j\notin J} h^{\pm}(q_i - q_j) \left(\prod_{l\in J} T_l^{\pm}\right) \prod_{i\in J,\ j\notin J} h^{\pm}(q_j - q_i). \tag{412}$$

Here $J \in \{1,\dots N\}$ and

$$h^{\pm}(q_i - q_j) = \left(\frac{\sin\frac{1}{2l}(q_i - q_j \pm i\nu)}{\sin\frac{1}{2l}(q_i - q_j)}\right)^{\frac{1}{2}}$$

and

$$T_k^{\pm} = e^{\frac{i}{l}\frac{\partial}{\partial q_k}}.$$

The classical analogue of (412) is defined as

$$S_{\pm 1} = \sum_{j=1}^{N} \prod_{i\neq j} \sqrt{1 - \frac{\sin^2(i\nu)}{\sin^2(q_i - q_j)}}\, e^{p_j k}.$$

One can show that

$$[\hat{S}_{\pm k}, \hat{S}_{\pm m}] = 0.$$

Now we consider $\Delta^{-1}\hat{S}_{\pm k}\Delta$. Denote $Q_j = e^{\frac{i}{l}q_j}$. Now for $t = e^{-\frac{\nu}{l}}$ define

$$h^{\pm}(q_i - q_j) = t^{\mp\frac{1}{2}}\left(\frac{t^{\pm 1}Q_i - Q_j}{Q_i - Q_j}\right)^{\frac{1}{2}}.$$

The action of the shift operator can be described by

$$T_k^{\pm} f(Q_1,\dots Q_N) = f(Q_1,\dots q^{\pm}Q_k \dots Q_n).$$

Then

$$S_{\pm k} = \sum_{|J|=k} \prod_{i\in J,\ j\notin J} \left(\frac{tQ_i - Q_j}{Q_i - Q_j}\right)^{\frac{1}{2}} \left(\prod_{l\in J} T_l^{+}\right) \left(\frac{t_j^{Q} - Q_i}{Q_j - Q_i}\right)^{\frac{1}{2}}.$$

When we conjugate the function with the ground state, we obtain an operator with symmetric polynomials being its eigenfunctions. The ground state in our theory is given by

$$\Delta = \left(\prod_{i\neq j}^{N} \prod_{k=0}^{\infty} \frac{Q_i - q^k Q_j}{Q_i - tq^k Q_j}\right)^{\frac{1}{2}} = \prod_{k=0}^{\infty} \Delta_k,$$

where

$$\Delta_k = \prod_{i\neq j}^{N}\left(\frac{Q_i - q^k Q_j}{Q_i - tq^k Q_j}\right)^{\frac{1}{2}}.$$

As we can see, $\Delta = \Delta_0\Delta_1\dots$ Hence, we can conjugate it by parts: first by Δ_0, then by Δ_1 and so on. After doing it, we obtain the McDonald operator

$$\hat{M}_k = \Delta^{-1}\hat{S}_k\Delta = t^{\frac{k(N-k)}{2}}\sum_{|J|=k}\prod_{i\in J, j\notin J}\frac{tQ_i - Q_j}{Q_i - Q_j}\prod_{l\in J}T_l^+.$$

The eigenfunctions of the McDonald operator are given by the symmetric polynomials. Define a set of the symmetric polynomials $P_\lambda(Q_1,\dots Q_N,q,t)$, q is a shift parameter,

$$P_\lambda(Q_1,\dots Q_N,q,t) = m_\lambda + \sum_{\mu<\lambda} u_{\mu\lambda}m_\mu.$$

One can define a scalar product by

$$< P_\lambda, P_\mu >= z_\lambda\delta_{\lambda\mu}\prod_{i=1}^{l(\lambda)}\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Here $z_\lambda = 1^{M_1}M_1!2^{M_2}M_2!\dots$ Here M_i is a multiplier of λ_i and

$$m_\lambda = \sum_{\sigma\in S_N} Q_{\sigma(1)}^{\lambda_1}\dots Q_{\sigma(N)}^{\lambda_N}.$$

For the McDonald polynomials we can write the spectral equation

$$\hat{M}_k P_k = h_k P_k,$$

where $h_k = \sum\limits_{j=2}^{N} q^{\lambda_j}t^{N-j}$.

Consider some examples. Let $N = 2$ (2 particles) and the Yung diagram ($\lambda_1 = \lambda_2 = 0$) is empty, $P = 1$, then

$$M_1' = \frac{tQ_1 - Q_2}{Q_1 - Q_2}T_1 + \frac{tQ_2 - Q_1}{Q_2 - Q_1}T_2.$$

The eigenvalues are $h_1 = t + 1$. Since $T_1 \cdot 1 = 1 = T_2 \cdot 1$, then

$$M_1' \cdot 1 = \frac{tQ_1 - Q_2}{Q_1 - Q_2} - \frac{tQ_2 - Q_1}{Q_1 - Q_2} = t + 1.$$

The next example is when the Yung's diagram consists of only one box: $\lambda_1 = 1, \lambda_2 = 0$

$$\square$$

Then $h_1 = tq + 1$ is the eigenvalue corresponding to $m_\lambda = Q_1^{\lambda_1}Q_2^{\lambda_2} + \text{Sym}$. Then $Q_1 + Q_2 = P$. Then we see that $T_1 P = qQ_1 + Q_2$ and $T_2 P = Q_1 + Q_2 q$,

$$\hat{M}_1' P = \frac{tQ_1 - Q_2}{Q_1 - Q_2}\cdot(qQ_1 + Q_2) + \frac{tQ_2 - Q_1}{Q_2 - Q_1}(Q_1 + Q_2 q).$$

Consider now the terms in front of the corresponding powers

$$q^1 : \frac{tQ_1 - Q_2}{Q_1 - Q_2}Q_1 - \frac{tQ_2 - Q_1}{Q_1 - Q_2}Q_2 = \frac{t(Q_1^2 - Q_2^2)}{Q - 1 - Q_2} = t(Q_1 + Q_2),$$

$$q^0 : \frac{tQ_1 - Q_2}{Q_1 - Q_2}Q_2 - \frac{tQ_2 - Q_1}{Q_2 - Q_1}Q_1 = Q_1 + Q_2.$$

Finally,

$$\hat{M}_1'(Q_1 + Q_2) = qt(Q_1 + Q_2) + (Q_1 + Q_2) = (tq + 1)(Q_1 + Q_2).$$

Now we wish to see the connection between this model and KZ-equations. Let us first recall the formula for the transfer matrix of the spin chain

$$\hat{t}(z) = \mathrm{tr}_0(R_{0,n}(z - z_n)R_{0,n-1}(z - z_{n-1} \dots R_{01}(z - z_1))g^{(n)}) = 1 \cdot \mathrm{tr}(g) + \eta \sum \frac{\hat{H}_j}{z - z_j}.$$

The transfer matrices commute

$$[\hat{t}(z), \hat{t}(w)] = 0.$$

The Hamiltonian is given by

$$\hat{H}_i = R_{i,i-1}(z_i - z_{i-1}) \dots R_{i,1}(z_i - z_1)g^{(1)}R_{in} \dots R_{i,i+1}.$$

The quantum KZ equations are defined as

$$\hat{T}_i|\Psi >= K^{(n)}|\Psi >, \tag{413}$$

where $\hat{T}_i = e^{\eta\hbar\partial_{x_i}}$ and $K_i^\hbar = R_{i,i-1}^\hbar(x_i - x_1 + \eta\hbar)g^{(i)} \otimes R_{ni}^{-1}(x_n - x_i) \dots R_{i+1,i}^{-1}(x_{i+1} - x_i)$. If we impose the unitarity condition

$$R_{12}(x)R_{21}(-x) = 1,$$

the expression becomes easier. We can put

$$\widetilde{R}_{ij} = \frac{x \cdot 1 + \eta P_{ij}}{z + \eta}.$$

Then

$$\hat{H}_i = \hat{K}_i^{(0)} \prod_{j \neq i} \frac{x_i - X_j + \eta}{x_i - x_j}.$$

Here $\hbar = 0$and shifts vanish. Note that

$$R_{12}(x) = 1 + \eta P_{12}$$

for the spin chain (simple pole at 0), for KZ equations the normalisation is different. Then

$$\widetilde{R}_{ij} = \frac{x + \eta}{x}R_{ij}.$$

Now multiply the (413) by $< \Omega|$. Remember that

$$\mathcal{H} = (\mathbb{C}^N)^{\otimes n} = \bigoplus_{\mu_1, \dots, \mu_N} \nu\left(\{M_k\}\right),$$

where N is a rank of spin chain. For instance, $N = 2$ corresponds to GL_2 XXX spin chain. Choose some basis $|J>$ in the sector of the Hilbert space, then

$$< \Omega| = \sum_J < J|,$$

$$< \Omega|P_{ij} =< \Omega|$$

then we can consider

$$\sum_{i=1}^{n} < \Omega|\hat{T}_i|\Psi >,$$

then

$$< \Omega|K_i^{\hbar}|\Psi >=< \Omega|K_i^{(0)}|\Psi >= \prod_{i \neq j} \frac{x_i - x_j + \eta}{x_i - x_j} < \Omega|\hat{H}_i|\Psi > .$$

After summing all the equations up, we will obtain the Hamiltonian. The result is as follows: consider $\Psi =< \Omega|\Psi >$, then

$$\sum_{i=1}^{n} \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j} e^{\eta\hbar\partial_{x_i}} \Psi =< \Omega| \sum_{i=1}^{n} \hat{H}_i|\Psi >= E\Psi,$$

where $E = \sum\limits_{a=1}^{N} M_a g_a$. Now one can consider a limit $\hbar \to 0$. Then we have, on the one hand, a spin chain, but, on the other, classical many-body system of the Ruseinaars or Calogero type. The parameter η is relativistic, x_i correspond to the positions of the particles. There is a quantum-classical duality $\dot{x}_i = h_i$ (eigenvalues of the quantum Hamiltonians). From now on we may no longer need a Bethe Ansatz — we can work with the many body sytem. One many-body system corresponds to a set of the spin chains (not to one) and there is a combinatorial rule to find the proper relation.

МЕХАНИКО-
МАТЕМАТИЧЕСКИЙ
ФАКУЛЬТЕТ
МГУ ИМЕНИ
М.В. ЛОМОНОСОВА

teach-in
ЛЕКЦИИ УЧЕНЫХ МГУ